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MATHEMATICS

RELIABILITY ESTIMATION OF WEIBULL LOMAX DISTRIBUTION VIA BAYESIAN APPROACH

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Weibull Lomax distribution is considered. Bayesian method of estimation is employed in order to estimate the reliability function of Weibull Pareto distribution by using non-informative and beta priors. In this paper, the Bayes estimators of the reliability function have been obtained under squared error, precautionary and entropy loss functions.

Keywords: Weibull Lomax distribution, reliability, Bayesian method, non-informative and beta priors, squared error, precautionary and entropy loss functions

1. INTRODUCTION

The Weibull-Lomax distribution has been proposed by Tahir *et al.* [1] They discussed the maximum likelihood estimation of the model parameters. The probability function $f(x;\theta)$ and distribution function $F(x;\theta)$ of Weibull Lomax distribution are respectively given by:

$$f(x;\theta)=\frac{ab\theta}{\lambda}\left[1+(x/\lambda)\right]^{ab-1}\left[1-\left(1+(x/\lambda)\right)^{-a}\right]^{b-1} e^{-\theta\left[\left(1+(x/\lambda)\right)^{a}-1\right]^{b}} \quad ; x>0. \tag{1}$$

$$F(x;\theta)=1-exp\left\{-\theta\left[\left(1+(x/\lambda)\right)^{a}-1\right]^{b}\right\} \quad ; x>0. \tag{2}$$

Let $R(t)$ denote the reliability function, that is, the probability that a system will survive a specified time t comes out to be:

$$R(t)=exp\left\{-\theta\left[\left(1+(t/\lambda)\right)^{a}-1\right]^{b}\right\} \quad ; t>0. \tag{3}$$

The instantaneous failure rate or hazard rate, h(t) is given by:

$$h(t)=\frac{ab\theta}{\lambda}\left[1+(t/\lambda)\right]^{ab-1}\left[1-\left(1+(t/\lambda)\right)^{-a}\right]^{b-1}. \tag{4}$$

From equation (1) and (3), we get:

$$f\left(x;R(t)\right)=\frac{ab\left(1+(x/\lambda)\right)^{ab-1}\left[1-\left(1+(x/\lambda)\right)^{-a}\right]^{b-1}}{\lambda\left[\left(1+(t/\lambda)\right)^{a}-1\right]^{b}}\left[-log\,R(t)\right]$$
$$\times\left[R(t)\right]^{\left[\left\{\left(1+(x/\lambda)\right)^{a}-1\right\}/\left\{\left(1+(t/\lambda)\right)^{a}-1\right\}\right]^{b}}\ ;\ 0<R(t)\le 1. \tag{5}$$

The joint density function or likelihood function of (5) is given by:

$$f\left(\underline{x}\,|\,R(t)\right)=\prod_{i=1}^{n}\left[\frac{ab\left(1+(x_i/\lambda)\right)^{ab-1}\left[1-\left(1+(x_i/\lambda)\right)^{-a}\right]^{b-1}}{\lambda\left[\left(1+(t/\lambda)\right)^{a}-1\right]^{b}}\right]$$
$$\times\left[-log\,R(t)\right]^{n}\left[R(t)\right]^{\sum_{i=1}^{n}\left[\left\{\left(1+(x_i/\lambda)\right)^{a}-1\right\}/\left\{\left(1+(t/\lambda)\right)^{a}-1\right\}\right]^{b}} \tag{6}$$

The log likelihood function is given by:

$$log\ f\left(\underline{x}\,|\,R(t)\right)=log\left(\prod_{i=1}^{n}\left[\frac{ab\left(1+(x_i/\lambda)\right)^{ab-1}\left[1-\left(1+(x_i/\lambda)\right)^{-a}\right]^{b-1}}{\lambda\left[\left(1+(t/\lambda)\right)^{a}-1\right]^{b}}\right]\right)$$
$$+n\,log\left[-log\,R(t)\right]+\left(\sum_{i=1}^{n}\left[\left\{\left(1+(x_i/\lambda)\right)^{a}-1\right\}/\left\{\left(1+(t/\lambda)\right)^{a}-1\right\}\right]^{b}\right)log\left[R(t)\right]. \tag{7}$$

Differentiating (7) with respect to R (t) and equating to zero, we get the maximum likelihood estimator of R (t) as:

$$\hat{R}(t)=exp\left[-n\left\{\left[\left(1+(t/\lambda)\right)^{a}-1\right]^{b}\Big/\sum_{i=1}^{n}\left[\left(1+(x_i/\lambda)\right)^{a}-1\right]^{b}\right\}\right]. \tag{8}$$

2. BAYESIAN METHOD OF ESTIMATION

The Bayesian estimation procedure has been developed generally under squared error loss function:

$$L\left(\hat{R}(t), R(t)\right) = \left(\hat{R}(t) - R(t)\right)^2 \tag{9}$$

where $\hat{R}(t)$ is an estimate of $R(t)$. The Bayes estimator under the above loss function, say $\hat{R}(t)_S$, is the posterior mean, *i.e.*,

$$\hat{R}(t)_S = E\left[R(t)\right]. \tag{10}$$

The squared error loss function is often used also because it does not lead extensive numerical computation but several authors (Zellner [2], Basu & Ebrahimi [3]) have recognized the inappropriateness of using symmetric loss function. Canfield [4] points out that its use may be inappropriate in the estimation of reliability function. Norstrom [5] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss function with quadratic loss function as a special case. A very useful and simple asymmetric precautionary loss function is:

$$L\left(\hat{R}(t), R(t)\right) = \frac{\left(\hat{R}(t) - R(t)\right)^2}{\hat{R}(t)}. \tag{11}$$

The Bayes estimator of $R(t)$ under precautionary loss function is denoted by $\hat{R}(t)_P$, and is obtained by solving the following equation:

$$\hat{R}(t)_P = \left[E\left(R(t)\right)^2\right]^{\frac{1}{2}}. \tag{12}$$

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\frac{\hat{R}(t)}{R(t)}$. In this case, Calabria and Pulcini [6] points out that a useful asymmetric loss function is the entropy loss $L(\delta) \propto \left[\delta^{p} - p\ log_{e}(\delta) - 1\right]$, where $\delta = \hat{R}(t) \big/ R(t)$, whose minimum occurs at $\hat{R}(t) = R(t)$ when $p > 0$, a positive error $\left(\hat{R}(t) > R(t)\right)$ causes more serious consequences than negative error, and *vice-versa*. For small $|\, p \,|$ value, the function is almost symmetric when both $\hat{R}(t)\, and\ R(t)$ are measured in a logarithmic scale, and approximately:

$$L(\delta) \propto \frac{p^{2}}{2}\left[log_{e}\hat{R}(t) - log_{e} R(t)\right]^{2}.$$

Also, the loss function $L(\delta)$ has been used in Dey *et al*. [7] and Dey and Liu [8], in the original form having $p = 1$. Thus $L(\delta)$ can be written as:

$$L(\delta) = b\left[\delta - log_{e}(\delta) - 1\right]; \quad b > 0. \tag{13}$$

The Bayes estimator of $R(t)$ under entropy loss function, denoted by $\hat{\theta}_{E}$, is obtained as:

$$\hat{R}(t)_{E} = \left[E\left(\frac{1}{R(t)}\right)\right]^{-1}. \tag{14}$$

For the situation where we have no prior information about $R(t)$, we may use non-informative prior distribution:

$$h_{1}(R(t)) = \frac{1}{R(t)\ logR(t)}; \quad 0 < R(t) \le 1. \tag{15}$$

The most widely used prior distribution for $R(t)$ is a beta distribution with parameters $\alpha, \beta > 0$, given by:

$$h_{2}(R(t)) = \frac{1}{B(\alpha,\beta)}\left[R(t)\right]^{\alpha-1}\left[1 - R(t)\right]^{\beta-1}; \quad 0 < R(t) \le 1. \tag{16}$$

3. BAYES ESTIMATORS OF $R(t)$ UNDER $h_1(R(t))$

Under $h_1(R(t))$, the posterior distribution is defined by:

$$f(R(t)|\underline{x}) = \frac{f(\underline{x}|R(t))h_1(R(t))}{\int_0^1 f(\underline{x}|R(t))h_1(R(t))dR(t)} \tag{17}$$

Substituting the values of $h_1(R(t))$ and $f(\underline{x}|R(t))$ from equations (15) and (6) in (17), we get:

$$f(R(t)|\underline{x}) = \frac{\left[\begin{array}{c}\prod_{i=1}^{n}\left[\dfrac{ab(1+(x_i/\lambda))^{ab-1}\left[1-(1+(x_i/\lambda))^{-a}\right]^{b-1}}{\lambda\left[(1+(t/\lambda))^{a}-1\right]^{b}}\right]\left[-log\, R(t)\right]^{n} \\ \times\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{(1+(x_i/\lambda))^{a}-1\right\}/\left\{(1+(t/\lambda))^{a}-1\right\}\right]^{b}\right)}\dfrac{1}{R(t)\ log\, R(t)}\end{array}\right]}{\int_0^1\left[\begin{array}{c}\prod_{i=1}^{n}\left[\dfrac{ab(1+(x_i/\lambda))^{ab-1}\left[1-(1+(x_i/\lambda))^{-a}\right]^{b-1}}{\lambda\left[(1+(t/\lambda))^{a}-1\right]^{b}}\right]\left[-log\, R(t)\right]^{n} \\ \times\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{(1+(x_i/\lambda))^{a}-1\right\}/\left\{(1+(t/\lambda))^{a}-1\right\}\right]^{b}\right)}\dfrac{1}{R(t)\ log\, R(t)}\end{array}\right]dR(t)}$$

$$= \frac{\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{(1+(x_i/\lambda))^{a}-1\right\}/\left\{(1+(t/\lambda))^{a}-1\right\}\right]^{b}\right)-1}\left[\text{-}log\, R(t)\right]^{n-1}}{\int_0^1\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{(1+(x_i/\lambda))^{a}-1\right\}/\left\{(1+(t/\lambda))^{a}-1\right\}\right]^{b}\right)-1}\left[\text{-}log\, R(t)\right]^{n-1}dR(t)}$$

or:

$$f(R(t)|\underline{x}) = \left[\frac{1}{\Gamma(n)}\left(\sum_{i=1}^{n}\left[\left\{(1+(x_i/\lambda))^{a}-1\right\}/\left\{(1+(t/\lambda))^{a}-1\right\}\right]^{b}\right)^{n}\right] \times\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{(1+(x_i/\lambda))^{a}-1\right\}/\left\{(1+(t/\lambda))^{a}-1\right\}\right]^{b}\right)-1}\left[\text{-}log\, R(t)\right]^{n-1} \tag{18}$$

Theorem 1. Assuming the squared error loss function, the Bayes estimate of $R(t)$ takes the form:

$$\hat{R}(t)_S = \left(1 + \frac{\left[\left(1+(t/\lambda)\right)^a - 1\right]^b}{\sum_{i=1}^{n}\left[\left(1+(x_i/\lambda)\right)^a - 1\right]^b}\right)^{-n} \tag{19}$$

Proof. From equation (10), on using (18):

$$\hat{R}(t)_S = E\left[R(t)\right]$$

$$= \int_0^1 R(t)\, f\left(R(t)\,|\,\underline{x}\right) dR(t)$$

$$= \int_0^1 R(t) \frac{1}{\Gamma(n)} \left(\sum_{i=1}^{n}\left[\left\{\left(1+(x_i/\lambda)\right)^a - 1\right\} \Big/ \left\{\left(1+(t/\lambda)\right)^a - 1\right\}\right]^b\right)^n$$

$$\times \left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{\left(1+(x_i/\lambda)\right)^a - 1\right\} / \left\{\left(1+(t/\lambda)\right)^a - 1\right\}\right]^b\right) - 1} \left[\text{-}log\, R(t)\right]^{n-1} dR(t)$$

$$= \frac{1}{\Gamma(n)} \left(\sum_{i=1}^{n}\left[\left\{\left(1+(x_i/\lambda)\right)^a - 1\right\} \Big/ \left\{\left(1+(t/\lambda)\right)^a - 1\right\}\right]^b\right)^n$$

$$\times \int_0^1 \left[R(t)\right]^{\sum_{i=1}^{n}\left[\left\{\left(1+(x_i/\lambda)\right)^a - 1\right\} / \left\{\left(1+(t/\lambda)\right)^a - 1\right\}\right]^b} \left[\text{-}log\, R(t)\right]^{n-1} dR(t)$$

$$= \frac{1}{\Gamma(n)} \left[\frac{\sum_{i=1}^{n}\left[\left(1+(x_i/\lambda)\right)^a - 1\right]^b}{\left[\left(1+(t/\lambda)\right)^a - 1\right]^b}\right]^n \times \Gamma(n) \left[\frac{\sum_{i=1}^{n}\left[\left(1+(x_i/\lambda)\right)^a - 1\right]^b}{\left[\left(1+(t/\lambda)\right)^a - 1\right]^b} + 1\right]^{-n}$$

or:

$$\hat{R}(t)_S = \left(1 + \frac{\left[\left(1+(t/\lambda)\right)^a - 1\right]^b}{\sum_{i=1}^{n}\left[\left(1+(x_i/\lambda)\right)^a - 1\right]^b}\right)^{-n}.$$

Theorem 2. Assuming the precautionary loss function, the Bayes estimate of $R(t)$ takes the form:

$$\hat{R}(t)_P = \left[1 + \frac{2\left[\left(1+(t/\lambda)\right)^a - 1\right]^b}{\sum_{i=1}^{n}\left[\left(1+(x_i/\lambda)\right)^a - 1\right]^b}\right]^{-\frac{n}{2}}. \tag{20}$$

Proof. From equation (12), on using (18):

$$\hat{R}(t)_P = \left[E\left(R(t)\right)^2\right]^{\frac{1}{2}}$$

$$= \left[\int_0^1 \left(R(t)\right)^2 f\left(R(t\,|\,\underline{x})\right) dR(t)\right]^{\frac{1}{2}}$$

$$= \left[\int_0^1 \left[\begin{array}{c}\left(R(t)\right)^2 \dfrac{1}{\Gamma(n)}\left(\sum_{i=1}^{n}\left[\left\{\left(1+(x_i/\lambda)\right)^a - 1\right\}\Big/\left\{\left(1+(t/\lambda)\right)^a - 1\right\}\right]^b\right)^n \\ \times\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{\left(1+(x_i/\lambda)\right)^a - 1\right\}/\left\{\left(1+(t/\lambda)\right)^a - 1\right\}\right]^b\right)-1}\left[\text{-}log\, R(t)\right]^{n-1}\end{array}\right] dR(t)\right]^{\frac{1}{2}}$$

$$= \left[\begin{array}{c}\dfrac{1}{\Gamma(n)}\left(\sum_{i=1}^{n}\left[\left\{\left(1+(x_i/\lambda)\right)^a - 1\right\}\Big/\left\{\left(1+(t/\lambda)\right)^a - 1\right\}\right]^b\right)^n \\ \times\displaystyle\int_0^1\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{\left(1+(x_i/\lambda)\right)^a - 1\right\}/\left\{\left(1+(t/\lambda)\right)^a - 1\right\}\right]^b\right)+1}\left[\text{-}log\, R(t)\right]^{n-1} dR(t)\end{array}\right]^{\frac{1}{2}}$$

$$= \left[\frac{\left(\sum_{i=1}^{n}\left[\left\{\left(1+(x_i/\lambda)\right)^a - 1\right\}\Big/\left\{\left(1+(t/\lambda)\right)^a - 1\right\}\right]^b\right)^n \Gamma(n)}{\Gamma(n)\left(\left(\sum_{i=1}^{n}\left[\left\{\left(1+(x_i/\lambda)\right)^a - 1\right\}\Big/\left\{\left(1+(t/\lambda)\right)^a - 1\right\}\right]^b\right)+2\right)^n}\right]^{\frac{1}{2}}$$

$$=\left[\frac{\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^a-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^a-1\right\}\right]^b\right)^n}{\left(\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^a-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^a-1\right\}\right]^b\right)+2\right)^n}\right]^{\frac{1}{2}}$$

or:
$$\hat{R}(t)_P=\left[1+\frac{2\left[\left(1+\left(t/\lambda\right)\right)^a-1\right]^b}{\sum_{i=1}^{n}\left[\left(1+\left(x_i/\lambda\right)\right)^a-1\right]^b}\right]^{-\frac{n}{2}}.$$

Theorem 3. Assuming the entropy loss function, the Bayes estimate of $R(t)$ takes the form:

$$\hat{R}(t)_E=\left[1-\frac{\left[\left(1+\left(t/\lambda\right)\right)^a-1\right]^b}{\sum_{i=1}^{n}\left[\left(1+\left(x_i/\lambda\right)\right)^a-1\right]^b}\right]^n \tag{21}$$

Proof. From equation (14), on using (18):

$$\hat{R}(t)_E=\left[E\left(\frac{1}{R(t)}\right)\right]^{-1}=\left[\int_0^1\frac{1}{R(t)}f\left(R(t\,|\,\underline{x})\right)dR(t)\right]^{-1}$$

$$=\left[\int_0^1\left[\begin{array}{l}\frac{1}{R(t)}\frac{1}{\Gamma(n)}\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^a-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^a-1\right\}\right]^b\right)^n\\ \quad\times\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^a-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^a-1\right\}\right]^b\right)-1}\left[\text{-}log\,R(t)\right]^{n-1}\end{array}\right]dR(t)\right]^{-1}$$

$$=\left[\begin{array}{l}\frac{1}{\Gamma(n)}\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^a-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^a-1\right\}\right]^b\right)^n\\ \times\int_0^1\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^a-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^a-1\right\}\right]^b\right)-2}\left[\text{-}log\,R(t)\right]^{n-1}dR(t)\end{array}\right]^{-1}$$

$$=\left[\frac{\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^a-1\right\}\Big/\left\{\left(1+\left(t/\lambda\right)\right)^a-1\right\}\right]^b\right)^n\Gamma(n)}{\Gamma(n)\left(\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^a-1\right\}\Big/\left\{\left(1+\left(t/\lambda\right)\right)^a-1\right\}\right]^b\right)-1\right)^n}\right]^{-1}$$

$$=\left[\frac{\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^a-1\right\}\Big/\left\{\left(1+\left(t/\lambda\right)\right)^a-1\right\}\right]^b\right)^n}{\left(\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^a-1\right\}\Big/\left\{\left(1+\left(t/\lambda\right)\right)^a-1\right\}\right]^b\right)-1\right)^n}\right]^{-1}$$

or: $$\hat{R}(t)_E=\left[1-\frac{\left[\left(1+\left(t/\lambda\right)\right)^a-1\right]^b}{\sum_{i=1}^{n}\left[\left(1+\left(x_i/\lambda\right)\right)^a-1\right]^b}\right]^n.$$

4. BAYES ESTIMATORS OF $R(t)$ UNDER $h_2\left(R(t)\right)$

Under $h_2\left(R(t)\right)$, the posterior distribution is defined by:

$$f\left(R(t)\,|\,\underline{x}\right)=\frac{f\left(\underline{x}\,|\,R(t)\right)h_2\left(R(t)\right)}{\int_0^1 f\left(\underline{x}\,|\,R(t)\right)h_2\left(R(t)\right)dR(t)} \tag{22}$$

Substituting the values of $h_2\left(R(t)\right)$ and $f\left(\underline{x}\,|\,R(t)\right)$ from equations (16) and (6) in (22), we get:

$$f\left(R(t)\,|\,\underline{x}\right)=\frac{\left[\begin{array}{l}\prod_{i=1}^{n}\left[\frac{ab\left(1+\left(x_i/\lambda\right)\right)^{ab-1}\left[1-\left(1+\left(x_i/\lambda\right)\right)^{-a}\right]^{b-1}}{\lambda\left[\left(1+\left(t/\lambda\right)\right)^{a}-1\right]^{b}}\right]\left[-\log R(t)\right]^{n}\\ \times\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^{a}-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^{a}-1\right\}\right]^{b}\right)}\frac{1}{B(\alpha,\beta)}\left[R(t)\right]^{\alpha-1}\left[1-R(t)\right]^{\beta-1}\end{array}\right]}{\int_0^1\left[\begin{array}{l}\prod_{i=1}^{n}\left[\frac{ab\left(1+\left(x_i/\lambda\right)\right)^{ab-1}\left[1-\left(1+\left(x_i/\lambda\right)\right)^{-a}\right]^{b-1}}{\lambda\left[\left(1+\left(t/\lambda\right)\right)^{a}-1\right]^{b}}\right]\left[-\log R(t)\right]^{n}\\ \times\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^{a}-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^{a}-1\right\}\right]^{b}\right)}\frac{1}{B(\alpha,\beta)}\left[R(t)\right]^{\alpha-1}\left[1-R(t)\right]^{\beta-1}\end{array}\right]dR(t)}$$

$$=\frac{\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^{a}-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^{a}-1\right\}\right]^{b}\right)+\alpha-1}\left[\text{-}\log R(t)\right]^{n}\left[1-R(t)\right]^{\beta-1}}{\int_0^1\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^{a}-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^{a}-1\right\}\right]^{b}\right)+\alpha-1}\left[\text{-}\log R(t)\right]^{n}\left[1-R(t)\right]^{\beta-1}dR(t)}$$

or:

$$f\left(R(t)\,|\,\underline{x}\right)=\frac{\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^{a}-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^{a}-1\right\}\right]^{b}\right)+\alpha-1}\left[\text{-}\log R(t)\right]^{n}\left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^{k}\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\frac{\left(1+\left(x_i/\lambda\right)\right)^{a}-1}{\left(1+\left(t/\lambda\right)\right)^{a}-1}\right]^{b}\right)+\alpha+k\right)^{n+1}\right]}\tag{23}$$

Theorem 4. Assuming the squared error loss function, the Bayes estimate of $R(t)$ takes the form:

$$\hat{R}(t)_S=\left[\frac{\sum_{k=0}^{\beta-1}(-1)^{k}\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^{a}-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^{a}-1\right\}\right]^{b}\right)+\alpha+1+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^{k}\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\left\{\left(1+\left(x_i/\lambda\right)\right)^{a}-1\right\}/\left\{\left(1+\left(t/\lambda\right)\right)^{a}-1\right\}\right]^{b}\right)+\alpha+k\right)^{n+1}}\right]\tag{24}$$

Proof. From equation (10), on using (23):

$$\hat{R}(t)_S = E\left[R(t)\right] = \int_0^1 R(t)\, f\left(R(t)\,|\,\underline{x}\right) dR(t)$$

$$= \int_0^1 R(t) \frac{\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{(1+(x_i/\lambda))^a-1\right\}\Big/\left\{(1+(t/\lambda))^a-1\right\}\right]^b\right)+\alpha-1}\left[\text{-}log\, R(t)\right]^n\left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\frac{(1+(x_i/\lambda))^a-1}{(1+(t/\lambda))^a-1}\right]^b\right)+\alpha+k\right)^{n+1}\right]} dR(t)$$

$$= \frac{\int_0^1\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\left\{(1+(x_i/\lambda))^a-1\right\}\Big/\left\{(1+(t/\lambda))^a-1\right\}\right]^b\right)+\alpha}\left[\text{-}log\, R(t)\right]^n\left[1-R(t)\right]^{\beta-1} dR(t)}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\left\{(1+(x_i/\lambda))^a-1\right\}\Big/\left\{(1+(t/\lambda))^a-1\right\}\right]^b\right)+\alpha+k\right)^{n+1}\right]}$$

or:

$$\hat{R}(t)_S = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\left\{(1+(x_i/\lambda))^a-1\right\}\Big/\left\{(1+(t/\lambda))^a-1\right\}\right]^b\right)+\alpha+1+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\left\{(1+(x_i/\lambda))^a-1\right\}\Big/\left\{(1+(t/\lambda))^a-1\right\}\right]^b\right)+\alpha+k\right)^{n+1}}\right].$$

Theorem 5. Assuming the precautionary loss function, the Bayes estimate of $R(t)$ takes the form:

$$\hat{R}(t)_P = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\frac{(1+(x_i/\lambda))^a-1}{(1+(t/\lambda))^a-1}\right]^b\right)+\alpha+2+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\frac{(1+(x_i/\lambda))^a-1}{(1+(t/\lambda))^a-1}\right]^b\right)+\alpha+k\right)^{n+1}}\right]^{\frac{1}{2}} \tag{25}$$

Proof. From equation (12), on using (23):

$$\hat{R}(t)_P = \left[E\left(R(t)\right)^2\right]^{\frac{1}{2}}$$

$$= \left[\int_0^1 \left(R(t)\right)^2 f\left(R(t\,|\,\underline{x})\right) dR(t)\right]^{\frac{1}{2}}$$

$$= \left[\int_0^1 \left(R(t)\right)^2 \frac{\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\frac{\left(1+\left(x_i/\lambda\right)\right)^a-1}{\left(1+\left(t/\lambda\right)\right)^a-1}\right]^b\right)+\alpha-1}\left[\text{-}log\,R(t)\right]^n\left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\frac{\left(1+\left(x_i/\lambda\right)\right)^a-1}{\left(1+\left(t/\lambda\right)\right)^a-1}\right]^b\right)+\alpha+k\right)^{n+1}\right]} dR(t)\right]^{\frac{1}{2}}$$

$$= \left[\frac{\int_0^1 \left[R(t)\right]^{\left(\sum_{i=1}^{n}\left[\frac{\left(1+\left(x_i/\lambda\right)\right)^a-1}{\left(1+\left(t/\lambda\right)\right)^a-1}\right]^b\right)+\alpha+1}\left[\text{-}log\,R(t)\right]^n\left[1-R(t)\right]^{\beta-1} dR(t)}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\frac{\left(1+\left(x_i/\lambda\right)\right)^a-1}{\left(1+\left(t/\lambda\right)\right)^a-1}\right]^b\right)+\alpha+k\right)^{n+1}\right]}\right]^{\frac{1}{2}}$$

or:

$$\hat{R}(t)_P = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\frac{\left(1+\left(x_i/\lambda\right)\right)^a-1}{\left(1+\left(t/\lambda\right)\right)^a-1}\right]^b\right)+\alpha+2+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left[\frac{\left(1+\left(x_i/\lambda\right)\right)^a-1}{\left(1+\left(t/\lambda\right)\right)^a-1}\right]^b\right)+\alpha+k\right)^{n+1}}\right]^{\frac{1}{2}}.$$

Theorem 6. Assuming the entropy loss function, the Bayes estimate of $R(t)$ takes the form:

$$\hat{R}(t)_E = \left[\frac{\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left[\frac{\left(1+(x_i/\lambda)\right)^a - 1}{\left(1+(t/\lambda)\right)^a - 1} \right]^b \right) + \alpha + k \right)^{n+1}}{\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left[\frac{\left(1+(x_i/\lambda)\right)^a - 1}{\left(1+(t/\lambda)\right)^a - 1} \right]^b \right) + \alpha - 1 + k \right)^{n+1}} \right] \tag{26}$$

Proof. From equation (14), on using (23):

$$\hat{R}(t)_E = \left[E\left(\frac{1}{R(t)} \right) \right]^{-1} = \left[\int_0^1 \frac{1}{R(t)} f\left(R(t \mid \underline{x}) \right) dR(t) \right]^{-1}$$

$$= \left[\int_0^1 \frac{1}{R(t)} \frac{\left[R(t)\right]^{\left(\sum_{i=1}^{n} \left[\left\{ \left(1+(x_i/\lambda)\right)^a - 1 \right\} / \left\{ \left(1+(t/\lambda)\right)^a - 1 \right\} \right]^b \right) + \alpha - 1} \left[\text{-}log\, R(t) \right]^n \left[1 - R(t) \right]^{\beta-1}}{\Gamma(n+1) \left[\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left[\frac{\left(1+(x_i/\lambda)\right)^a - 1}{\left(1+(t/\lambda)\right)^a - 1} \right]^b \right) + \alpha + k \right)^{n+1} \right]} dR(t) \right]^{-1}$$

$$= \left[\frac{\int_0^1 \left[R(t)\right]^{\left(\sum_{i=1}^{n} \left[\left\{ \left(1+(x_i/\lambda)\right)^a - 1 \right\} / \left\{ \left(1+(t/\lambda)\right)^a - 1 \right\} \right]^b \right) + \alpha - 2} \left[\text{-}log\, R(t) \right]^n \left[1 - R(t) \right]^{\beta-1} dR(t)}{\Gamma(n+1) \left[\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left[\left\{ \left(1+(x_i/\lambda)\right)^a - 1 \right\} / \left\{ \left(1+(t/\lambda)\right)^a - 1 \right\} \right]^b \right) + \alpha + k \right)^{n+1} \right]} \right]^{-1}$$

$$= \left[\frac{\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left[\left\{ \left(1+(x_i/\lambda)\right)^a - 1 \right\} / \left\{ \left(1+(t/\lambda)\right)^a - 1 \right\} \right]^b \right) + \alpha - 1 + k \right)^{n+1}}{\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left[\left\{ \left(1+(x_i/\lambda)\right)^a - 1 \right\} / \left\{ \left(1+(t/\lambda)\right)^a - 1 \right\} \right]^b \right) + \alpha + k \right)^{n+1}} \right]^{-1}$$

or: $$\hat{R}(t)_E = \left[\frac{\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left[\frac{\left(1+\left(x_i/\lambda\right)\right)^a - 1}{\left(1+\left(t/\lambda\right)\right)^a - 1} \right]^b \right) + \alpha + k \right)^{n+1}}{\sum_{k=0}^{\beta-1} (-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^{n} \left[\frac{\left(1+\left(x_i/\lambda\right)\right)^a - 1}{\left(1+\left(t/\lambda\right)\right)^a - 1} \right]^b \right) + \alpha - 1 + k \right)^{n+1}} \right].$$

5. CONCLUSIONS

We have obtained a number of Bayes estimators of reliability function $R(t)$ of Weibull Lomax distribution. In equations (19), (20), and (21), we have obtained the Bayes estimators by using non-informative prior and in equations (24), (25), and (26), under beta prior. From the above equations, it is clear that the Bayes estimators of $R(t)$ depend upon the parameters of the prior distribution. In this case, the risk function and corresponding Bayes risks do not exist.

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