Memoirs of the Scientific Sections of the Romanian Academy
Tome XLVI, 2023

PHYSICS

THE PHYSICAL BASIS OF PROCOPIU'S QUANTIZATION

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The procedures of quantization are the same in the case of matter as in the case of light. They involve specific invariants to the scale change, and specific statistics describing that scale change in physics. A previous part of this work [35] explores the general mathematics of the physics involved in the problem of quantization. The present part deepens the similitude between the procedures of quantization in light and matter, by proving the identity of the resonators serving for quantization from a thermodynamical point of view. They are dipoles: electric in the case of light – Planck's resonators, representing the matter – and magnetic in the case of matter – Procopiu's resonators, representing the light. Mathematical details are exposed, and some physical consequences in observance thereof are explored.

Keywords: Louis de Broglie's ray, Planck's resonator, Procopiu's resonator, resonator's structure, universal optical medium, instanton, Ampère current element

There are more things in heaven and earth, Horatio,
than are dreamt of in your philosophy.
Hamlet, Prince of Denmark

1. INTRODUCTION

Procopiu's quantization procedure is the only quantization procedure in the case of matter that reproduces the Planck's archetypal procedure from the case of light [35]. The essential differences between the two procedures of quantization stay in details of realization: (1) the structure of the quantum (universal constant, in the case of Planck, *vs*. invariant, in the case of Procopiu), and (2) the associated statistics (discrete, in the case of Planck, *vs*. continuous, in the case of Procopiu). However, the nature of the two essential items of this kind of quantization procedure is the same in the two cases, indicating the differentiae of such a concept: the quantum is a *mathematical invariant* connected with a dynamics just like the classical

Newtonian force, and the associated statistics is that of ensembles described by probability densities with *quadratic variance function*, invented by Max Planck in order to serve the quantization. These two defining differentiae of the concept of quantization procedure initiated by Max Planck are, by and large, *not* of physical nature. In this work, however, we intend to go deeper into the parallel between the quantization of light and the quantization of matter, by showing that it goes beyond non-physical criteria. There is a more detailed parallelism between the two procedures, and it regards the very physical structure of *the resonator* invented by Max Planck which, besides its definition as an instanton given by us previously (see [35], §4.4), needs a little theoretical explanation in order to be suitably understood and applied for the case of matter.

Our ideas in approaching this analogy came largely from applying the scale transition perspective to a description of the physical structure of matter. We were thereby led to a specific comprehension of the modern asymptotic freedom concept that, from the natural philosophy perspective, was made possible only by the quantization procedure. From this viewpoint, the particles realizing the cohesion of matter are photons, insofar as they need to be described like photons: they must be conceived as free particles, at least formally speaking. Particularly illuminating, at least for what we have to say here anyway, are the 2004 Nobel lectures of the recognized forerunners of the theory of asymptotic freedom – and their different works, of course, serving for the foundation of this physical concept – especially David Gross' Nobel Lecture and the presentation slides [21].

In order to unveil from the very beginning the natural-philosophical point of view, we start with the observation that Max Planck defined the concept of resonator in order to account for the equilibrium temperature of the radiation in a physical enclosure containing matter and light. Even though Planck presented his choice as just an incidental one, the physics ever since tells us another story: his resonator is the most convenient concept that could ever serve the purpose of quantization. In a word, the resonator is a *necessity*, not just an *incident*. Without this choice we are not able to construct a thermodynamics of radiation based on the electromagnetic theory of light. And, as we intend to show here, there is, indeed, a more general theory of radiation necessarily asking for the Planck's concept of resonator: the electromagnetic theory is just a particular case.

Fact is that, according to Kirchhoff's laws of radiation, a thermal equilibrium can be established in a Wien-Lummer enclosure [53] containing radiation and matter. This equilibrium is essential in defining a temperature of radiation according to the laws of thermodynamics. However, the Kirchhoff's laws of radiation are only phenomenological: they do not require details regarding the physical structure of the matter used in describing this thermal equilibrium. They simply ask for matter as a category, respecting only the thermodynamical prescriptions. Therefore, according to these laws, we can think freely of *any physical structure* of the matter in thermodynamical equilibrium with radiation in an enclosure, in order to be able to define the temperature of radiation.

We can think, for instance, of an ideal gas, as Wily Wien once did (see [52] for a closer documentation), on the occasion of establishing the radiation law bearing his name, *i.e.*, the *Wien*'*s radiation law*. This is, in our opinion, the first sign that the Kirchhoff's laws are not quite so general in order to satisfy the thermodynamical requirements of thermal equilibrium. More to the point, their generality should not stand upon the idea of arbitrariness of the matter structure in equilibrium with radiation, but upon the idea of *determining that structure in order to realize such an equilibrium*. For the Wien's case in point, the ideal gas is the only one liable of having an absolute temperature connected to a sufficient statistics – namely, the average kinetic energy of the molecular chaos – by its very definition. Then, and only then, if a thermodynamical equilibrium is established in an enclosure containing gas and radiation, one can talk about the constant temperature of the matter inside enclosure, which is measured by the absolute temperature of the gas. Only at this point, therefore, can we apply the thermodynamical reasons, to the effect that the absolute temperature should also be the radiation's temperature, considering, of course, the light as a physical system submitted to the same thermodynamic laws as the matter. This is just the general idea, but in order to get real regarding an equilibrium of the two physical "substructures" – light and matter – of the content of a Wien-Lummer enclosure, we need to consider the interaction between them, and this fact brings complications.

It is on this occasion when Max Planck used the fact that the Kirchhoff's laws *do not require* a structure for the matter in the enclosure, *as an advantage*. To wit, he translated the lack of requirement into the liberty of an invention, *physically decided* based on the fact that the light is an electromagnetic phenomenon. Indeed, Wien's ideal gas considerations, on the occasion of establishing his law of radiation, were conducted in such a way that they disregarded an important physical point of concern: the mechanism of interaction of the matter with radiation. Indeed, in his reasoning, Wien resorted heavily on the classical 'analogy', "if we may say so, between the kinetic energy of molecules and the energy of light, as calculated from its intensity". This allowed him to skip the details of the mechanism of interaction that would be able to explain physically the thermodynamical equilibrium.

On the other hand, Planck's choice for the fundamental structure of the matter in equilibrium with radiation – the resonator – is an electric vibrating dipole, instead of just the simple material point of the ideal gas. It was deliberately chosen to fill in for such a missing point: the physical details of the interaction of the matter with radiation should be explicit, inasmuch as the radiation is electromagnetic. Indeed, it was known from Hertz's electromagnetic theory ([27], pp. 137 *ff*) that such a dipole can absorb and emit electromagnetic light. Thus, based on this natural phenomenon, a thermal equilibrium can be *physically* established inside a Wien-Lummer cavity, which is liable to be theoretically described in terms of the electrical dipoles interacting with radiation, by a kind of statistical theory. And, to this end, Max Planck even constructed a special statistics for describing the physics of this kind of equilibrium: the statistical theory of ensembles characterized by probabilities having densities with variance function quadratic in their mean values (see [35], §2.3). Reproducing, for conformity and documentation, Planck's own words, the ideas above are expressed as follows:

> … This law (*Kirchhoff's*, *a*/*n*) states that *a vacuum* completely enclosed by reflecting walls, in which any emitting and absorbing bodies are scattered in any arrangement whatever, *assumes in the course of time the stationary state of black radiation*, which is *completely determined* by *one parameter* only, namely, *the temperature*, and in particular does not depend on *the number*, the *nature*, and the *arrangement* of the material bodies present. Hence, for the investigation of the properties of the state of black radiation, *the nature of the bodies which are assumed to be in the vacuum is perfectly immaterial*. In fact, it does not even matter *whether such bodies really exist somewhere in nature*, provided their existence and their properties are consistent with the laws of thermodynamics and electrodynamics. If, for *any special arbitrary assumption* regarding the nature and arrangement of emitting and absorbing systems, we can find a state of radiation in the surrounding vacuum *which is distinguished by absolute stability*, *this state can be no other than that of black radiation*.
>
> Since, *according to this law*, *we are free to choose any system whatever*, we now select from all possible emitting and absorbing systems the simplest *conceivable* one, namely, one consisting of a large number N of similar *stationary oscillators*, each *consisting of two poles*, *charged with equal quantities of electricity of opposite sign*, which may move relatively to each other on a *fixed straight line*, *the axis of the oscillator*.
>
> It is true that it would be more general and *in closer accord with the conditions in nature* to assume the *vibrations* to be those of *an oscillator consisting of two poles*, each of which *has three degrees of freedom* of motion instead of one, *i.e*., to assume the vibrations *as taking place in space instead of in a straight line only*. Nevertheless we may, *according to the fundamental principle stated above*, restrict ourselves from the beginning to the treatment of one single component, *without fear of any essential loss of generality of the conclusions we have in view*. ([39], pp. 135 – 136; *emphasis added*, *n*/*a*)

Taking up the issues presented here by starting from the end one of this excerpt, the history of physics proved, quite contrarily, that the generality was lost, in fact, and even in an important aspect at that: the fundamental structures of the matter in equilibrium with radiation are not arbitrary, but *they must be dipoles*. This is one of the essential conclusion of the discovery of the modern asymptotic freedom [21]. One can say that the generality can be preserved, indeed, but only from a dynamical point of view and, moreover, even by a special formulation of dynamics. This formulation contains the equilibrium in a specific way, according to the *idea of force characterizing a statics*, once voiced by Eugene Wigner on the occasion of a proposal of some particular foundations of the wave mechanics [54]. However, in order to take heed of such an idea, we need to learn some more lessons from the physics of light along the lines of Planck's procedure of quantization. And because Procopiu's procedure of quantization is the only one going along those very lines, but specifically in the case of matter not light, we may be allowed to go back to a presentation of the essentials of the theory of Procopiu quantization in the most notable form of a counterpart of Wien-Lummer cavity [53], involving the forces directly. However, these are to be conceived in a special optical medium that generalizes the electromagnetic vacuum, but points out to the dipole as a *necessary fundamental structure* in physics, as required by the asymptotic freedom.

The general natural-philosophical grounds for our approach in constructing a counterpart of Planck's ideas expressed in the previous excerpt, but with application to the case of matter, are as follows. First, notice that in the very Planck's expression, the content of a Wien-Lummer enclosure in the problem of radiation is, so to speak, 'a piece of vacuum'. Then, nothing appears as more natural than taking the fundamental unit of matter imagined by Planck as necessary in solving the problem of radiation – the Planck's resonator – as "a piece of vacuum having two electric charges at the ends". This is, indeed, just the definition of an electric dipole in the Katz's natural philosophy of charge (see [34], §3.1); for conformity, see also the original [30]). Then, according to the same natural philosophy, the correspondent in matter of the Planck's resonator should be, simply, «a piece of matter having two magnetic charges at the ends», *i.e.*, a magnetic dipole, which we propose to call a *Procopiu*'*s resonator*, in view of the fact that the Procopiu's quantization is a quantization in the case of matter that completely parallels that of Planck's from the case of light (see [35], *passim*). The hard part of such an analogy would then be not conceiving the resonator because this is readily available to our intellect, but conceiving the *equivalent of a Wien-Lummer cavity* "completely enclosing matter". However, according to the very same natural philosophy of Katz for charges, this concept seems to have been exercised by the human intellect for ages. Indeed, by "duality", as it were, we can think that this should be "a *matter* enclosed by reflecting walls". If these reflecting walls are taken, by the very same "duality principle", as made from "vacuum" – replicating the reflecting walls of Planck, which are made from "matter", obviously – we get the well-known image, largely utilized in theoretical physics, of *an isolated extended particle*, standing alone in a universe.

One can say that this is the kind of physical particle that can be described by the first quantization procedure ever. It can be defined, according to the precepts of special relativity, as a genuine 'instanton' (see [35], Chapter 4, especially §4.4): a collection of simultaneous events described by a **sl** (2, R) Riemannian manifold. The present work gives, by and large, the essential mathematics helping to understand the physics of such a natural philosophy, but from the perspective of the modern asymptotic freedom concept.

2. THE RESONATOR AS A FUNDAMENTAL PHYSICAL STRUCTURE

Regardless the category whose physical structure it explains – and talking of category we have in mind a kind of Kantian use of the concept, meaning specifically matter and light here, taken as two *different categories* – the resonator has, according to Katz's natural philosophy of charges, the very same fundamental physical structure: *a dipole*; *electric*, representing the matter in the case of light, respectively *magnetic*, representing the light in the case of matter. And while we are along this path of mending the philosophical meaning of categories, let us notice that the vacuum is a category, too: it is taken here as meaning *the absence of matter*, according to the physical definition of this concept. This objectively means matter interpreted by particles having Newtonian forces between them, in order to be conceivable as ensembles of particles in equilibrium. From this point of view it is important to notice that matter and light are two *opposite* categories: the physics, mainly that of the last century, has established that they go into one another when disappearing. Neither of them goes into vacuum. The dipole will be presented here, mathematically, as a metric structure regardless of the idea of interpretation: a property of the geodesics of a special Riemann manifold of positive curvature. The general incentives for such a presentation come, by and large, from the Louis de Broglie's theory of optical ray (see [5], especially [5b] and [5c]) serving for completion of the Fresnel's physical theory of light, by proving its harmony with the quantum concepts (see [34], Chapter 2, §2.1).

2.1. THE GENERAL THEORY OF AN OPTICAL RAY

The first problem to be solved in the de Broglie's order of things physical – for, this is, indeed, an order set forth by an analysis of the work of Louis de Broglie (see [34], *passim*) – is that of some theoretical requirements for the physical description of a light ray. To start with, one needs to know the equation of progression of the phenomenon of light along an optical ray, *i.e.* a mathematical description of the propagation phenomenon from the point of view of the ray theory. The mathematical point of view in the natural-philosophical requirements on propagation of light is usually represented *via* a local displacement in an arbitrary direction from a point along the ray, which thus decides the ray path. The mathematics is, again usually, handled by the so-called *Euler-Lagrange equations* describing the displacement according to the idea that the real path of the light is corresponding to an extremum of the *optical path*, defined as (see, *e.g.*, [48,49]; these are the works we follow here, anyway):

$$I \overset{def}{=} \int n(x,y,z)\cdot\sqrt{\langle dx|dx\rangle} \tag{2.1.1}$$

where $(ds)^2 \equiv \langle dx|dx\rangle$ is the square of arclength of the corresponding *geometrical path*, *i.e.* of the path in the empty space hosting the optical medium of *refraction index* $n(\boldsymbol{x})$, assumed to be Euclidean. The variational problem associated with the integral from equation (2.1.1) – the Fermat's principle – provides the differential 'equation of motion along the ray', where the *geometrical path* length is playing the part of 'time' of motion. From this perspective, therefore, the motion represents a propagation, whose time is represented by the geometrical path length from a position to another, as in classical optics, where the light is assumed to propagate through free space. The differential equation is:

$$\frac{d}{ds}\left(n(\boldsymbol{x})\frac{d}{ds}|x\rangle\right) = \nabla n(\boldsymbol{x}) \tag{2.1.2}$$

This demands a little explanation from our part, mainly regarding the *ket* notation.

In this kind of optical problems, the geometry is dealing with *coordinates of location* only. Thus, what is meant by the symbol $|x\rangle$, as well as by $\boldsymbol{x}$, is a set of three coordinates in space, locating the positions in a Cartesian reference frame. Only, the Dirac's notation suggests an algebraic realization of the vector as a 3×1 matrix, *i.e.*, a matrix with three lines and a column. In the cases where the reference frame is unique in space, like in path optics, there is no difference in meaning between the two notations. Such a difference occurs only when the reference frame changes along the path, and it is important for the global geometry involved in the optics of light, in that the geometrical quantities of physical interest are the *torsions*, not the *curvatures* as usual [8].

For now, though, coming back to our trail of discussion here, regarding the classical optics, after working on the algebraical expansion of the equation (2.1.2), we end up with the differential equation:

$$n(\boldsymbol{x})|x''\rangle + (\nabla n(\boldsymbol{x})\cdot|x'\rangle)|x'\rangle = \nabla n(\boldsymbol{x}) \tag{2.1.3}$$

Here, the accent means differentiation upon s, and a dot between vectors means the regular dot-product of the vectors. Assuming a functional form of the refraction index of the medium, will give us the properties of propagation in that physical medium – if it exists at all – described by that refraction index. To wit, let us choose a refraction index having the functional form:

$$n(\boldsymbol{x}) = \left(1 + \langle x|x\rangle\right)^{-1} \quad \therefore \quad \nabla n(\boldsymbol{x}) = -2\left(1 + \langle x|x\rangle\right)^{-2}|x\rangle \tag{2.1.4}$$

where $\langle x|x\rangle$ is the sum of squares of the coordinates along the path of light. The coordinates are taken here in a form scaled with reference to some gauge lengths, in order to maintain the spirit of Fresnel's physical theory of light, whereby the coordinates are to be considered pure numbers, so that the notation makes sense for now, from a mathematical point of view. The equation (2.1.3) then becomes:

$$\left(1 + \langle x|x\rangle\right)|x''\rangle - 2\langle x|x'\rangle|x'\rangle + 2|x\rangle = |0\rangle \tag{2.1.5}$$

Using the definition of the elementary arclength of the path in terms of coordinates, which we take a priori as Cartesian, obviously in an – again, assumed – Euclidean background space, we have:

$$(ds)^2 = \langle dx|dx\rangle = \langle x'|x'\rangle(ds)^2 \quad \therefore \quad \langle x'|x'\rangle = 1 \tag{2.1.6}$$

Then, by differentiating the last equality, we get the relations:

$$\langle x'|x''\rangle = 0, \quad \langle x''|x''\rangle + \langle x'|x'''\rangle = 0 \tag{2.1.7}$$

that can be used in order to conclude on the equation (2.1.5). First, by differentiating the very equation (2.1.5), we have:

$$(1 + \langle x|x\rangle)|x'''\rangle - 2\langle x|x''\rangle|x'\rangle = |0\rangle \tag{2.1.8}$$

whence dot-multiplying this by $|x''\rangle$ and using the first relation from (2.1.7), we get:

$$\langle x''|x'''\rangle = 0 \quad \therefore \quad \langle x''|x''\rangle = const \tag{2.1.9}$$

Geometrically, this means that the curvature of the ray path should be constant for this kind of continuum, for the curvature of a geometrical path is in fact measured by the second derivative of the coordinates along that path. Returning then to (2.1.5) once again, but this time for dot-multiplying it by $|x''\rangle$ directly, while using the first of the results (2.1.7) and the result (2.1.9), gives:

$$\frac{2\langle x|x''\rangle}{1+\langle x|x\rangle}=-\frac{1}{R^2},\quad \langle x''|x''\rangle\equiv\frac{1}{R^2} \tag{2.1.10}$$

where R is a non-dimensional constant, suggesting, again, the necessity of a gauge length in defining the curvature. As we shall see, this observation is of tremendous importance in deciding upon the definition of one of the most important concepts of physics, the frequency, based on the phenomenon of holography. Bluntly put, the frequency is, indeed, defined as a measure of curvature. For now, though, just inserting the first relation (2.1.10) into equation (2.1.8) produces one final equation:

$$|x'''\rangle+\frac{1}{R^2}|x'\rangle=|0\rangle \tag{2.1.11}$$

This is the "genuine optical occurrence", as it were, of a third order linear differential equation which is liable to describe the concept of Hooke's light ray (see [35], §3.2, equation (3.2.14); for conformity see [28], pp. 55–65). However, more important for a theory of matter to which that old concept is referring, (2.1.11) describes the dynamical necessities of the regularization theory for the Kepler motion [see [35], §3.4, equations (3.4.9) and (3.4.18)]. This last instance comes down to considering the space occupied by the center of force in the Kepler dynamical problem *as an optical medium of this sort*.

While these tasks will be gradually accomplished as we go along with our work, for now we have a general observation that needs to be made in order to properly guide the work itself. Namely, the *optical medium* described by the refraction index from equation (2.1.4) should be considered a Riemannian manifold which turns out to be of *finite volume* having positive curvature. Thus, it can play the part of a Wien-Lummer cavity in the thermodynamics of light studies: we have, therefore, a theoretical model of this experimental device, described as a Riemannian space. Recall, once again, that such a cavity was, and still is in fact, the device of experimental study of the light from a thermodynamic point of view, serving in obtaining the right laws of radiation. Modern high-tech researches point out to the important fact that the universe we inhabit can be taken as such a device [14].

Going for some details along the line pinpointed in the previous excerpt from Planck, the Planck's 'vacuum' is represented here by a transparent continuum, having the refraction index given by the equation (2.1.4), counting geometrically as a Riemannian manifold of positive curvature. Indeed, the elementary optical path of this medium is conform-Euclidean, assuming that $\langle dx|dx\rangle$ is Euclidean, as we did. When introducing two suitable parameters a, b in order to characterize the Euclidean shape of the piece of matter representing this medium, the regular geometric form of the metric becomes [6]:

$$(ds)^2 = 4a^2b^2 \frac{\langle dx | dx \rangle}{(b^2 + \langle x | x \rangle)^2} \tag{2.1.12}$$

Here we have applied the equation (2.1.1), used to establish the optical path definition, in its infinitesimal instance, of course, so that ds is the *elementary optical path*.

This is the metric of the realm called *Maxwell fish-eye*, and is especially interesting for us because it has circles as geodesics, with the property of an electric dipole's lines of force: all the geodesics passing through any point of the realm in any one of its two-dimensional sections, also pass through a point which is its transform by reciprocal radii with respect to an appropriate sphere. That is, optically speaking, the Maxwell fish-eye is a perfect device in which all light rays through a point have the *properties of the lines of force of an electric dipole*: circles passing through two fixed points representing the locations of the two component charges of the dipole (see [48], Chapter IV; [49], §§1.6, 1.7 and 2.9).

Should the necessity occur to operate an interpretation here, in the manner required by the wave theory of light [11], it obviously needs to be accomplished by particles having *electric charges*. However, these electric charges can have any values: they are not necessarily quantized. Case in point, they must have the Lorentz property, but in its utmost generality. To wit, in order to acquire a charge of opposite sign, a certain position from such a medium needs to be *replicated by inversion* with respect to a certain, locally spherical surface (see [32]; §§57 and 67; see also [34], §3.2 for details). That spherical surface is, according to Lorentz, the only 'bearer' of zero charge particles. In general, we can have here the property of a physical lens, as it were, characteristic to a portion of a surface, which can be differential, as well as fractal.

Let us show some details of these statements, first because, physically speaking, the theory contains the fundamental structure required by Planck's quantization procedure just naturally, but also because we need to be fairly familiar with the details of the procedure in view of its application in the theory of embeddings (we follow here [6], §73). These details involve the space embedding into a four-dimensional Euclidean manifold. Again, as we shall see here, this four-dimensional manifold is of essence in physics, in general. For a good guidance on the topic, so much the better as this guidance is offered in connection with classical non-Euclidean geometries, we recommend the exquisite work of Ruben Aldrovandi and José Geraldo Pereira on Geometrical Physics, especially Chapter 23 of that work [1]. As, further on, the embedding procedure involves the *stereographic projection*, which is of essence in constructing the counterpart of Planck's resonator in matter – the Procopiu's resonator, as we would like to call it – one may need a previous accommodation with this kind of projection. We recommend a geometrically thorough presentation of the stereographic projection method – analytic as well as synthetic geometry – which is made in the booklet [43]. Now, back to our line of discourse.

According to Constantin Carathéodory, the parameters a and b, that we have introduced previously, have the following meaning: the metric (2.1.12) is the metric of a three-dimensional manifold in a four-dimensional space, analytically represented by a four-dimensional Euclidean sphere of radius a, projected stereographically onto a three-dimensional Euclidean space at the distance b from the center of the projection. This can be shown by the following analytical procedure. Start with the observation that the equation of a Euclidean four-sphere in Cartesian coordinates ξ, η, ζ, τ is, by analogy with the three-dimensional case:

$$\xi^2 + \eta^2 + \zeta^2 + \tau^2 = a^2 \tag{2.1.13}$$

and the three-dimensional stereographic projection on one of its Euclidean tangent hyperplanes, from a point located at the distance b with respect to that hyperplane, is achieved by the formulas:

$$\frac{\xi}{x} = \frac{\eta}{y} = \frac{\zeta}{z} = \frac{\tau + a}{b} =: n \tag{2.1.14}$$

Introducing these coordinates in (2.1.13), we get an equation that can be solved right away, giving two values of n:

$$n = 0, \quad n = \frac{2ab}{b^2 + r^2} \tag{2.1.15}$$

Here $r^2 \equiv \langle x|x\rangle$ is the Euclidean norm of the position vector of the projected point from the tangent hyperplane. The first one of these values, $\tau = -a$, corresponds to the "south pole" of the hypersphere (2.1.13) – the "north pole", $\tau = a$, being the point of contact of the hyperplane (x, y, z) with the hypersphere – where the correspondence realized by equation (2.1.14) is singular. On the other hand, though, the second one of the values (2.1.15) corresponds to the projection of the current point of coordinates (ξ, η, ζ, τ), onto the "north pole" hyperplane, $\tau = a$, thus helping in representing the current point by a point in the "tangent" Euclidean space in coordinates (x, y, z). According to equation (2.1.14), this representation is provided by the formulas:

$$\xi = a\frac{2bx}{b^2 + r^2}, \quad \eta = a\frac{2by}{b^2 + r^2}, \quad \zeta = a\frac{2bz}{b^2 + r^2}, \quad \tau = a\frac{b^2 - r^2}{b^2 + r^2} \tag{2.1.16}$$

which can be readily solved for (x, y, z), and provide the Cartesian coordinates by the following ratios:

$$x = b\frac{\xi}{a+\tau}, \quad y = b\frac{\eta}{a+\tau}, \quad z = b\frac{\zeta}{a+\tau}, \quad r^2 = b^2\frac{a-\tau}{a+\tau} \tag{2.1.17}$$

Now, using these last two equations, we can construct the four-dimensional Euclidean elementary distance of the ambient space, as an Euclidean metric:

$$(ds)^2 = (d\xi)^2 + (d\eta)^2 + (d\zeta)^2 + (d\tau)^2 \tag{2.1.18}$$

which turns out to be the metric (2.1.12).

Going a little bit ahead of us here, we see these mathematical results the following way: the Maxwell fish-eye is an optical medium describing the matter in a three-dimensional Euclidean space where the light dwells. The matter in this space is itself a Riemannian manifold, having the metric (2.1.18), which is conformal with the Euclidean metric in three-space of our experience, as in equation (2.1.12). The problem is not what the three-space represents, because we know this from that experience, but what the coordinates (ξ, η, ζ, τ) are, and an answer presents itself right away: *they are charges*. This is a story first told to us by the geodesics of the conformal metric (2.1.12), which are lines characterizing the field of dipoles: either electric or magnetic. On the other hand, any two of the four coordinates (ξ, η, ζ, τ) can be associated with each other, in order to give either the square of an electric charge or the square of a magnetic charge according to Katz's natural philosophy (see [34], §3.1). The *association* is, a priori, a stochastic process and, as we shall show here, has everything in common with the stochastic type of processes once imagined by Carlton Frederick for the metric tensor of the spacetime [16]. So, we may say that the equation (2.1.13) represents here, by and large, an electromagnetic continuum split into charges by the procedure of embedding a three-dimensional Euclidean manifold.

This is a 'device', realized, in the case of light, by a Wien-Lummer matter cavity enclosing light and matter – this last category in the form of a physical structure made of Planck's resonators – in thermal equilibrium. In the case of matter, on the other hand, it should be realized by the 'dual' of this device, as it were: a vacuum-made cavity containing the matter to be quantized – a category that cannot have but a physical structure made of Procopiu resonators – in thermal equilibrium with light, which is a category that cannot have but a physical structure made out of dipoles, whose nature remains yet to be established. In any case, details aside, the quantization procedure must be that of Planck. It pays to notice the different roles played by light in the two situations: as we shall see, this is the reason for the fact that we have today the concept of Yang-Mills fields.

2.2. THE WIEN-LUMMER ENCLOSURE FOR MATTER

We will work here on some details of an example in three space coordinates, in order to get the grip on some well-known cases, which serve theoretically just for guiding purposes: afterwards, though, we can frame easier the previous four-dimensional case of the charges, which is of the same nature. Thus, we have a sphere centered in origin and radius R:

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = R^2 \tag{2.2.1}$$

to be projected onto the upper tangent plane $\xi_3 = R$ (the so-called north pole) from its center. If (x, y, R) are the coordinates of the point in plane upon which the point of coordinates (ξ_1, ξ_2, ξ_3) of the sphere is projected from origin, then this projection is described by the system of equations:

$$\frac{\xi_1}{x} = \frac{\xi_2}{y} = \frac{\xi_3}{R} =: \lambda \tag{2.2.2}$$

where λ is a parameter. Now, if the Euclidean metric of this continuum reproduces the signature of the quadratic form (2.2.1), then in terms of the coordinates of the plane we can write it as:

$$\begin{aligned}\sum_k (d\xi_k)^2 = (x^2 + y^2 + R^2)(d\lambda)^2 + \lambda^2 \left\{(dx)^2 + (dy)^2\right\} \\ +2\lambda d\lambda(xdx + ydy)\end{aligned} \tag{2.2.3}$$

Using (2.2.1) and (2.2.2) for calculating λ, we get:

$$\lambda^2 = \frac{R^2}{x^2 + y^2 + R^2} \quad \therefore \quad 2\lambda d\lambda = -2R^2 \frac{xdx + ydy}{(x^2 + y^2 + R^2)^2} \tag{2.2.4}$$

so that the metric (2.2.3) becomes:

$$\sum_k (d\xi_k)^2 = R^2 \left\{ \frac{(dx)^2 + (dy)^2}{x^2 + y^2 + R^2} - \left(\frac{xdx + ydy}{x^2 + y^2 + R^2} \right)^2 \right\} \tag{2.2.5}$$

Is this truly a metric in the geometrical sense, *i.e.* the elementary distance measure in space? The answer is affirmative: it is, indeed, the infinitesimal Euclidean distance in the three-dimensional space, calculated with the so-called *Laguerre's formula*, involving the logarithm of a cross-ratio. It is, obviously, realized as a metric in the two-dimensional case, *i.e.* on a surface. Let us get into some details.

To wit, if we denote by X a point in this space, then a coordinate representation is given by a *triple of numbers* representing the point in the sense of Cartan: memorize them somehow, and then carry them everywhere and realize the position in any place *via* an adequate reference frame [8]. A slight change in notation seems in order here, to the effect that the lower indices will be assigned to points, rather than to coordinates, which are to be taken as Cartesian coordinates:

$$X \stackrel{def}{=} (x, y, z) \tag{2.2.6}$$

Note, in this association, that X should not necessarily be taken as a vector: it is just a triple of numbers. This means that we shall build the geometry based on the properties of the quadratic form from the left hand side of equation (2.2.1), using, however, the properties of this quadratic form as we know them from the regular geometry:

$$(X, X) \equiv x^2 + y^2 + z^2 \tag{2.2.7}$$

Thus, the condition to have X as a real point in space is $(X, X) > 0$, even if this quantity is unspecified by the metric idea of a sphere in space, or something like that. By contrast, the condition $(X, X) < 0$ defines, from a geometrical point of view some purely imaginary points. However, from a physical point of view, such points can be only 'partially' imaginary, so to speak. The physical interpretation depends on the condition $(X, X) = 0$, and this can always make sense in physics, pending a condition of quantization. For instance, it can represent the condition of equilibrium of Newtonian forces within the static ensembles of particles serving for interpretation. Then stochastic processes can be defined in order to assimilate the fundamental physical quantities generating the three forces with lengths, serving to realize the Cartanian program [8]. The stochastic processes are defined by specific Lewis-Lutzky invariants, and thus they realize the necessary memory serving for accomplishing the program. In this specific case only the mass can be imaginary, and in microcosmos, where the charges prevail by their Newtonian forces it is, indeed, *always* imaginary (see for details [34], §§3.1 and 4.3).

In order to construct an absolute geometry based on these considerations (see for details and discussion [31]) we take the quadratic form (2.2.7) as a norm for the points in our space of points. It induces an internal multiplication of points (a dot-product, as it were) by the *polarization* process:

$$(X_1, X_2) \overset{def}{=} x_1x_2 + y_1y_2 + z_1z_2 \tag{2.2.8}$$

with an obvious correspondence between indices of points and indices of the corresponding coordinates. This dot-product, entirely analogous with the classical Euclidean dot-product, helps us in characterizing a straight line in space, which is the essential concept necessary in constructing a metric. The straight line joining two points X_1 and X_2 is, like in the regular Euclidean geometry, the locus of points:

$$X = \lambda X_1 + \mu X_2 \tag{2.2.9}$$

with λ and μ variable numbers representing the homogeneous parameters of points along the line. This straight line intersects the absolute $(X, X) = 0$ in two points, having the homogeneous parameters partially determined – meaning: up to an arbitrary factor – by the quadratic equation:

$$(X, X) \equiv \lambda^2 (X_1, X_1) + 2\lambda\mu (X_1, X_2) + \mu^2 (X_2, X_2) = 0 \tag{2.2.10}$$

That is, we can determine only the ratios of these two parameters – the *non-homogeneous coordinates* along the line – as the roots of this equation, *viz*.:

$$t \equiv \frac{\lambda}{\mu} = -\frac{(X_1, X_2)}{(X_1, X_1)} \pm \frac{\sqrt{(X_1, X_2)^2 - (X_1, X_1)\cdot(X_2, X_2)}}{(X_1, X_1)} \tag{2.2.11}$$

As it turns out, these two ratios are enough for our purpose of building a metric of the space.

Indeed, in geometry, a metric is, in fact, the *distance between two infinitesimally close* points, so that what we need first is to define a *distance* between points. Now, the quantity that reduces to the distance between two points in the Euclidean case, turns out to be the *cross ratio* of four points on a straight line: two of these points are fixed and used as a reference frame on the line, while any other pair of points is the current pair of points between which we calculate the distance. With reference the the straight line defined by equation (2.2.9), the two points having the parameters from equation (2.2.11) can be taken as the reference frame. Then the distance between X_1 and X_2 is, up to a numerical factor, the *logarithm of this cross ratio* (the so-called *Laguerre*'*s formula*). For, given two points $X_{1,2}$, the straight line joining them contains all points of the form $X = tX_1 + X_2$, forming the continuum whose geometrical form is that line. In order to define the distance between the two points, we can choose arbitrarily two other points, $X_{3,4}$ say, to play the part of the reference frame on the straight line. Then the cross ratio of points on line is simply defined as the cross ratio of the corresponding non-homogeneous parameters t. So, we have:

$$(X_1, X_2; X_3, X_4) \overset{def}{=} \frac{t_1 - t_3}{t_1 - t_4} : \frac{t_2 - t_3}{t_2 - t_4} \tag{2.2.12}$$

The Laguerre distance is simply *proportional* to the logarithm of this quantity, the coefficient of proportionality being a universal constant for a given geometry. It depends, of course, on the second pair of points, which is also our choice, and therefore may be deemed as subjective on occasions – thus suggesting a possible ambiguity – but this ambiguity can be substantially reduced if we refer the construction to the absolute of space: this is a sphere, and a straight line in space always intersects a sphere in two points. In this case, we notice first that, according to equation (2.2.9), the parameter t has the values $t_2 = 0$ for the point X_2 (*i.e.* $\lambda = 0$), and correspondingly, $t_1 = \infty$ for the point X_1 (*i.e.* $\mu = 0$). In view of this, the cross ratio (2.2.12) takes the simple form of a ratio:

$$(X_1, X_2; X_3, X_4) = \frac{t_4}{t_3} \tag{2.2.13}$$

upon which our choice for the two points X_3 and X_4 reveals the advantage of allowing a *standardization*, as it were, of this construction. Namely, disregarding the algebraical nature of the two numbers $t_{3,4}$, one can say that every pair of points in space has a corresponding pair of points on the absolute, these being the points where the corresponding straight line passing through those points intersects the absolute. These are *real* if the straight line meets the absolute, *identical* if the straight line is tangent to absolute, and *complex* if it does not touch the absolute. If the two points X_1 and X_2 are both inside the absolute, then the numbers $t_{3,4}$ must be real, no question about that. Thus, the corresponding parameters $t_{3,4}$ are then given by the two ratios from equation (2.2.11), so that equation (2.2.13) becomes:

$$(X_1, X_2; X_3, X_4) = \frac{(X_1, X_2) + \sqrt{(X_1, X_2)^2 - (X_1, X_1)\cdot(X_2, X_2)}}{(X_1, X_2) - \sqrt{(X_1, X_2)^2 - (X_1, X_1)\cdot(X_2, X_2)}} \tag{2.2.14}$$

This ratio, however, is a complex of unit modulus so it cannot serve the intended purpose, which requires reality of the distance. The conclusion can be ascertained from the fact that the quantity under the sign of square root is always negative for real vectors in the Euclidean space. Nevertheless, according to Felix

Klein, even with this cross-ratio, we can still construct a differential version of the distance by Cayley's method, *viz*. a metric of space [31]. Indeed, the distance according to Laguerre's formula is only *proportional* to the logarithm of the cross ratio, and therefore it involves an arbitrary constant. The logarithm of the cross ratio from equation (2.2.14) is a purely imaginary complex number, so that, if we choose the proportionality constant as an imaginary complex number the things are in order. Thus, the Laguerre distance given *via* the logarithm of the cross ratio (2.2.13) can be represented by the distance given *via* the logarithm of cross ratio (2.2.14), because the ratio of the two expressions involved in equation (2.2.14) is a purely imaginary complex number, and we are at liberty to choose an imaginary number as the constant defining the Laguerre distance.

Assuming, therefore, that in order to define the metric the two points X_1 and X_2 are infinitesimally close $X_1 = X$, $X_2 = X + dX$, just like in the regular Euclidean geometry, we can calculate the necessary factors in equation (2.2.14) as:

$$(X_1,X_2)^2 - (X_1,X_1)\cdot(X_2,X_2) = (X,dX)^2 - (X,X)\cdot(dX,dX) \qquad (2.2.15)$$

Now, in the *real domain*, we can accept that the quantity $(X,dX)/(X,X)$ is an infinitesimal quantity of the first order, while $(dX,dX)/(X,X)$ is an infinitesimal quantity of the second order. Thus, the cross ratio (2.2.14) can be expanded and, to first infinitesimal order it is:

$$(X_1,X_2;X_3,X_4) = 1 + 2i\sqrt{\frac{(dX,dX)}{(X,X)} - \left(\frac{(X,dX)}{(X,X)}\right)^2} \qquad (2.2.16)$$

The logarithm of this quantity is, to the same infinitesimal order, the second term from the right hand side, which is, of course, a purely imaginary number, as we just said. Then, we can set things in order by Klein's recipe: multiply the logarithm with a purely imaginary constant quantity, *iR* say, in view of the fact that the metric *per se* is defined up to an arbitrary scale factor. Thus the Cayley-Klein – or absolute – metric of this geometry can be finally written in the form:

$$\left(\frac{ds}{R}\right)^2 = \frac{(dX,dX)}{(X,X)} - \left(\frac{(X,dX)}{(X,X)}\right)^2 \qquad (2.2.17)$$

with R an arbitrary real quantity. This equation is a regularly considered form of the Cayley-Klein metric, with reference to any absolute of space. It turns out that this expression is also valid in *larger conditions of space definition*: *complex points, general definition of the absolute as a quadric in this space, etc*. Dan Barbilian, to mention a notable case, used it for the cases where (X,X) is a homogeneous polynomial of *arbitrary* degree – a quantic, in algebraic phraseology – thus generalizing the metric (2.2.17) even further [2].

However, as long as the absolute is a quadric – *i.e.*, a general surface specified by an equation quadratic in the coordinates – using the properties of the dot and cross products of the real vectors in space, the metric (2.2.17) can be written in the form:

$$\left(\frac{ds}{R}\right)^2 = \frac{(X \wedge dX, X \wedge dX)}{(X,X)^2} \tag{2.2.18}$$

with

$$X \wedge dX \overset{def}{=} (ydz - zdy, zdx - xdz, xdy - ydx)$$

Notice that this metric reduces to that from equation (2.2.5), given by projection, for $z = \pm R$. Therefore, the previous results – that is, the ones that we can get *via* the method of projection – are also obtainable as absolute geometrical results, just by assuming that one of the coordinates – usually z – is constant: this time, though, the constant is not quite arbitrary, but needs to have specifically the value R. In such a case, we have:

$$X \wedge dX \overset{def}{=} (-Rdy, Rdx, xdy - ydx) \tag{2.2.19}$$

and, if we apply to this the formula (2.2.18), we get the result:

$$\left(\frac{ds}{R}\right)^2 = R^2 \frac{(dx)^2 + (dy)^2}{(x^2 + y^2 + R^2)^2} + \left(\frac{xdy - ydx}{x^2 + y^2 + R^2}\right)^2 \tag{2.2.20}$$

This result, no question, coincides with the one from equation (2.2.5) up to a factor, but reveals an interesting position of the metric of a *Maxwell fish-eye* (2.1.12), which is the three-dimensional extension of (2.2.20). This hints to the universality of such a metric, at least from a physical point of view. Discussing, however, on the two-dimensional case in hand, if we work on the last term of the expression from the right hand side of equation (2.2.20) we can write it in the form:

$$\frac{xdy - ydx}{x^2 + y^2 + R^2} = \frac{r^2 d\theta}{r^2 + R^2}, \quad x = r\cos\theta, \quad y = r\sin\theta \tag{2.2.21}$$

revealing a great advantage in the cases where the Kepler's law of the areas is valid in the plane (x, y). However, regardless of such an occurrence, the equation (2.2.20) can then be written in the form:

$$\left(\frac{ds}{R}\right)^2 = R^2\frac{(dx)^2+(dy)^2}{(x^2+y^2+R^2)^2}+\tanh^4\phi\cdot(d\theta)^2 \quad r = R\sinh\phi \quad (2.2.22)$$

showing that, in cases where ϕ is constant, the Cayley-Klein metric basically differs by only the square of an exact differential from the Maxwell fish-eye metric. Therefore, in particular, this may be the case if have $\tanh^2\phi{\cdot}d\theta =$ *constant times a differential*, which can be seen as a second of Kepler laws, as we said, defining the time scale in the sense of regularization theory (see [35], §3.4). Such a case will be discussed as we go along with our developments in this work. The transcription (2.2.22) has only the purpose of revealing the position of the Maxwell fish-eye metric in context, nothing else. For, if we homogenize the notation from equation (2.2.22), extending it also to the first term from the right hand side, we get the final form of the absolute metric as:

$$\left(\frac{ds}{R}\right)^2 = \frac{(d\phi)^2}{\cosh^2\phi}+\tanh^2\phi\cdot(d\theta)^2 \quad (2.2.23)$$

which, by itself, is liable to explain the Langevin statistics used in realizing Procopiu's procedure of quantization (see [35], §2.4 for details).

It is, in this connection, worth disclosing right away the usefulness of the expression (2.2.23), if for nothing else, just for fostering the casual reader's curiosity. Assume that, for some reason, the parameter θ is constant indeed. We can realize the importance of such an occurrence in case this parameter is connected with a time scale change, as in the second of Kepler's laws. Then, a reason presents itself immediately for such an occurrence, based on physical facts: the time "freezes", as it were, and we have to deal with an enclosure containing *simultaneous* events. In other words, in this case, we have to deal with an *instanton* sui generis: a piece of matter, made of simultaneously existing physical structures. The metric (2.2.23) then reduces to an exact differential:

$$\frac{ds}{R} = \frac{d\phi}{\cosh\phi} \quad (2.2.24)$$

representing the elementary probability of a Hyperbolic Secant type ([35], §2.4). This type of probabilities is connected with the "magnetic" Langevin statistics which is of essence in realizing the Procopiu's quantization in matter, along the same lines of realization as those of the archetypal Planck's quantization. The *correlation function* of their probability densities is the partition function of the Langevin distribution [*loc. cit. ante*, equations (2.3.7) and (2.3.10)]:

$$\frac{\phi}{\sinh\phi} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{d\varphi}{\cosh\phi \cdot \cosh(\phi - \varphi)} \tag{2.2.25}$$

Therefore, according to Katz's natural philosophy of charges, such an instanton can be described as a matter structure in equilibrium with light – this one being described by an ensemble of Procopiu resonators – isolated within a Wien-Lummer cavity 'made' of vacuum. One can say that the Cayley-Klein, or absolute geometry is the geometrical image of this physical situation, and this concept has a real advantage: the geometrical crossing of the absolute can be *naturally* viewed as a representation of the modern physical concept of *vacuum tunneling* [29].

2.3. THE FREQUENCY: A GENERAL DEFINITION OF HOLOGRAPHY

The basis of Louis de Broglie's result regarding the correlation between the amplitude of an optical signal and the density of the optical medium supporting it, in his case of interpretation of light (see [34], §2.1), is the fact that the optical amplitude of the wave equivalent with a particle is independent on time, and that the optical signal has a perfectly determined optical frequency, with a phase which is function *only of space coordinates*:

$$u(\boldsymbol{x},t) = A(\boldsymbol{x})e^{i(\omega t+\phi(\boldsymbol{x}))}, \quad \nabla^2 A = 0 \tag{2.3.1}$$

[*loc. cit. ante*, equations (2.1.2) and (2.1.16)]. A case may be made then, based, however, on historical facts only, that the phenomenology of light involves *four* phenomena in order to construct the physics along the classical concept of the light ray. For, the classical light ray requires interpretation, and the interpretation needs a concept of particle, available to optics only by the idea of Louis de Broglie.

Let us restate the problem again, this time just in order to create a perspective. The physical description of light is commonly presented as having started with a single one of the phenomena characteristic to light. For instance, the initial properties of light were described based on *reflection*. Adding *refraction* led the intellect to appreciate the crystal physics, and made possible a quantitative characterization of Hooke's idea of periodicity of the light phenomenon, by the concept of wavelength and associated frequency. This last concept helped in characterizing the light when the addition of the *diffraction* phenomenon to the phenomenology of light became critical, in the times of Fresnel. More importantly though, the diffraction in the phenomenology of light led to the *discovery* of the fourth phenomenon of this specific experience: *holography*.

Indeed, if the diffraction was a known phenomenon – basically meant to be known in order to be avoided in the classical theory of light rays – we certainly cannot say the same about holography: the *man was completely unaware of it* before quantum theory surfaced for our intellect, so it was practically discovered. However, today we are forced to recognize that this phenomenon is, indeed, part and parcel of the phenomenology of light. Then we seem to be entitled to ask: what definition of the frequency should be used when starting with the holographic phenomenon instead of diffraction, or reflection, or refraction, in the physical description of the light? In answering this question, we have a "crossword clue", as it were: certainly, the classical concept of frequency used the idea of wavelength in order to be defined, and this, in turn, was extracted from experiments with crystals, involving therefore the reflection and refraction phenomena in *finite spaces*. On the other hand, the idea of a hologram came to being also inspired by the physics of crystals [17]. This suggests that the concepts to be targeted by starting with holography in the physics of light are the *wavelength* and the *frequency*. However, the modern concept of light is, as a rule, theoretically explained based on the idea of harmonic oscillator, whereby the measurements results are represented by signals, mathematically thought of as functions of time. Within this circumstance, the *concept of frequency* can be understood from the perspective of holography, along the following lines.

Like the classical *velocity*, that can be assigned only *via* the uniform motion of a material point, the *frequency* can only be assigned *via* the idea of a periodic motion. This means a spatially finite motion of a material point, which, therefore, cannot be free. Indeed, the physical prototype of a periodic motion – counting as the analogous of a free particle for the uniform motion, as it were – is the harmonic oscillator, mathematically described by an ordinary second-order differential equation:

$$\ddot{x}(t)+\omega_0^2 x(t)=0 \tag{2.3.2}$$

Thus, assigning a frequency to a *signal that can be represented as a periodic function of time* is a particularly simple task here, for it is provided by the above equation in the form:

$$\omega_0^2 \overset{def}{=} -\frac{\ddot{x}(t)}{x(t)} \tag{2.3.3}$$

suggesting a curvature-connected to the function $x(t)$. This definition hardly satisfies the idea of uniqueness of the frequency. Besides, the overwhelming majority of physical cases of signals are, in fact, not so simple, in order to be represented directly as harmonic oscillators: a complex signal asks, for instance, to be modeled by a Fourier series of virtually an infinity of harmonic oscillators of different frequencies, in order to be physically described. As a matter of fact, this is the reason why the assumption of a direct time dependence of the coordinates created so much trouble along the history, up to the point of a change in physics. To wit, it is sufficient to recall that the idea of Fourier series imposed an overturn of the established values of our intellect, by the emergence of the quantum mechanics [26].

In the case of light, though, the *phenomenon of holography* could not enter the phenomenology but only after admitting that the time dependence of the coordinates *has to be mediated by a phase*. Mention should be made of the significant *circumstance* that the realization of importance of this decision – namely that the time dependence should be mediated by a phase – came long after the de Broglie's realization of quantization in matter, and usually counts, for our intellect, as the most important consequence of the quantization procedure. However, the first incentives of the construction of a hologram are referring precisely to the concept of phase, *not the frequency*. Quoting:

> It is customary to explain this (*the Bragg's method of reconstruction of a lattice by diffraction, the source of Dennis Gabor's inspiration in devising his method*, *a/n*) by saying that the *diffraction diagrams contain information on the intensities only*, *but not on the phases*. The formulation is somewhat unlucky, as it suggests at once that *since the phases are unobservables*, this state of affairs *must be accepted*. In fact, not only that part of the phase which is unobservable drops out of conventional diffraction patterns, but also the part which corresponds to geometrical and optical properties of the object, and which in principle *could be determined by comparison with a standard reference wave*. It was this consideration which led me finally to the new method. ([18], *our emphasis*, *a/n*)

That 'standard reference wave' had to have a *different phase*, but *the very same frequency*: with the expression of Gabor himself, it has to be 'monochromatic coherent'. Without further ado about it, we see in the excerpt above the necessity of *intervention of the phase in the expression of the amplitude*: for, certainly, the measurements of light cannot be made but by the mediation of amplitude, once they are always based on the measurements of intensity. Then the basic principle of holography can be simply described mathematically as follows: it is the phenomenon occurring when we try to associate a time dependence to a physical process depending on time, however, *not in an obvious periodic way*, but implicitly, through the phase. The association of a frequency with light phenomenon then involves an *ensemble of phases*, which, from an optical point of view, means an ensemble of waves carrying the same information, in particular information about the same object.

In order to make this statement more graspable, let us describe the way this association is done mathematically. In the general case, when suspecting a periodic behavior of the phenomenon, we can try to model it by an equation representing an *a priori* periodic form. This is certainly the general case of modeling *via* periodic functions, which comprises the oscillator as a particular. A typical signal of this type is:

$$q(t) = A(t)e^{i\theta(t)} \tag{2.3.4}$$

which involves a phase θ and an amplitude A, both arbitrary functions of time. This process is, indeed, *a priori* periodical, *i.e.* periodic in the trigonometric sense, however not in time directly, as it has not an obvious frequency associated to it, as a periodic process of the oscillator kind has. If there is a frequency involved here, as Gabor's idea of standard reference wave asks for, it can only be exhibited if $q(t)$ represents a periodic motion of the kind described by equation (2.3.2). Assuming, therefore, such an equation for the signal $q(t)$, leads us to some identifications:

$$\ddot{q} + \omega_0^2 q = 0 \quad \therefore \quad \begin{array}{l} \dfrac{\ddot{A}}{A} + \omega_0^2 = \dot{\theta}^2 \\ 2\dfrac{\dot{A}}{A} + \dfrac{\ddot{\theta}}{\dot{\theta}} = 0 \end{array} \tag{2.3.5}$$

The second one of these equations from the right hand side here, gives right away:

$$A^2\dot{\theta} = const \tag{2.3.6}$$

which is a kind of Kepler's second law, known as the *area law*, and suggesting a periodic motion for the amplitude itself, when compared with the details of the Kepler problem. Of course, here this means that we shall need a kind of interpretation of this amplitude, and such an interpretation involves the classical idea of free… oscillator. That this is the case can be shown right away, since in these conditions the first of the equations (2.3.5) gives an Ermakov-Pinney equation for the amplitude:

$$\ddot{A} + \omega_0^2 A = \frac{R_0^2}{A^3} \tag{2.3.7}$$

where R_0 is a real constant. The connection with the periodic motion *per se* is then the following. Let A be the composite amplitude of a two-dimensional harmonic oscillator, described by a quadratic form in the partial amplitudes of component signals varying in time according to the equation of $q(t)$ from (2.3.5), *i.e.*, in particular:

$$A^2 = A_1^2 + A_2^2, \quad \ddot{A}_1 + \omega_0^2 A_1 = 0, \quad \ddot{A}_2 + \omega_0^2 A_2 = 0 \tag{2.3.8}$$

This amplitude satisfies the equation (2.3.7) with R_0 the constant from (2.3.6). Thus, the frequency ω_0 is asociated to the components of the vector $|A\rangle$ in an obvious way, inasmuch as they are oscillators. In these conditions, though, one can calculate right away that the square of amplitude – that is, the *intensity of signal* in optical terms – from equation (2.3.4) satisfies a linear third-order differential equation of known type:

$$\frac{d^3}{dt^3} A^2 + 4\omega_0^2 \frac{d}{dt} A^2 = 0 \tag{2.3.9}$$

Comparing this equation with the one of a ray from (2.1.11), a conclusion imposes by itself, namely that Louis de Broglie was right after all: the equation characterizing an optical ray is referring, indeed, to the square of the amplitude of an optical signal. Then, because the square of the amplitude of recorded signal is, according to de Broglie, the numerical density necessary for an incidental interpretation, the equation (2.3.9) should also be taken as an equation for that density.

It may appear that, with this conclusion, we are rushing in a little, 'where angels fear to tread', as they say. For once, the kind of ray described by a refraction index (2.1.4), which asks for an equation like (2.1.11) or (2.3.9), may not be universal, at least not to the same degree as the equation (2.3.4) for the mathematical model of a signal appears to be. The optical medium, described by a particular refraction index, as given in equation (2.1.4), may be very particular indeed. However, it is worth recalling that this kind of 'particular' is just mathematical here: from a physical point of view it may prove to be universal. A warning sign on this issue is the existence of the *Hanbury Brown-Twiss effect*: there are *intensity correlations* of the rays issuing from the same distant source of light [23,24]. Indeed, the square of the amplitude means, as we repeated quite a few times by now, an intensity in the optical realm. And, if an equation like (2.3.9) proves to be universal according to the general mathematical structure of a signal, then we can take that the medium of refraction index (2.1.4) is that necessary all-pervading medium of the classical ether type, support of every phenomenon in the world we inhabit. Anyway, at least we have guidance in our proceedings. To wit: we need to follow the idea of the meaning of a refraction index as the one suggested by this ray optics, and then, more importantly, to follow the track of an equation like (2.1.11) or (2.3.9). It is particularly important to know if such an equation appears anywhere else in physics at all, and in what conditions. As we have shown previously ([35], Chapter 3, §3.4), this equation is of essence in the regularization theory of Kepler motion. One thus can say that it is of essence in the problem of interpretation, securing the invariance to the scale transition of this interpretation.

A major problem still remains to be solved here, though, for it is directly connected to the equation (2.3.8), which, in turn, is conditioning any result declared thus far: how can we define the frequency in a proper way, that is, in such a way as to include the phenomenon of holography from the very beginning?! A sound solution imposes by itself through the *idea of coherence*, and can be obtained using the 'Kepler's second law' (2.3.6), which seems to be an apt universal mathematical fact, endorsed by the theory of regularization ([34], §3.4). Taking, therefore, for the amplitude as a function of phase, the definition provided by the Kepler's law (2.3.6), will be consistent with the *holographic principle* defined according to Dennis Gabor's ideas. For once, this definition would mean that *the time variation of phase must be physically recognizable in the intensity of a certain wave*. Then, proceeding just mathematically, we are able to transform the Keplerian condition (2.3.6) into a second-order differential equation for the amplitude of the complex signal:

$$A = \frac{C}{\sqrt{\dot{\theta}}} \quad \therefore \quad \ddot{A} + \frac{1}{2}\{\theta, t\} A = 0 \tag{2.3.10}$$

where C is a constant, and the notation:

$$\{\theta, t\} \stackrel{def}{=} \frac{d}{dt}\left(\frac{\ddot{\theta}}{\dot{\theta}}\right) - \frac{1}{2}\left(\frac{\ddot{\theta}}{\dot{\theta}}\right)^2 \tag{2.3.11}$$

represents the *Schwarzian derivative* of the phase with respect to time. In equation (2.3.10) the definition of a frequency is conspicuous, by comparison with the equation of motion of a simple harmonic oscillator (2.3.3). Indeed, *this defines the frequency in terms of the phase* of the general signal (2.3.4) by the equation:

$$\{q, t\} = 2w_0^2 \tag{2.3.12}$$

which allows for a plus of mathematical precision in formulating the holographic principle. For once, the Schwarzian derivative is a curvature [15], but let us show what we mean by this kind of precision, in some specific details.

Everything revolves now around the definition of the Schwarzian derivative [see, for relevant details and a comprehensive presentation of this operation [37], Chapter 5, §§X, XI, XII]. One property is striking in this definition, of which we shall make much use in this work: any solution of the equation (2.3.12) *is defined up* *to a homographic transformation*. This would mean that the manifold of solutions of equation (2.3.12) is three-dimensional, not in the sense of the linear superposition rule, though, but in the sense that it can be *surveyed* by locating its points with three parameters. In the superposition rules' phrasing, we rather have here a *nonlinear superposition rule* with three basic solutions of the equation [see [7], §§2, 3, especially equations (3.51–53)]. More precisely, knowing three solutions of the equation (2.3.12), a fourth one can be found right away, without any integration, because it must have a constant cross ratio with those three. In order to prove this statement, we use the general relation of transformation of the Schwarzian [see [37], especially Chapter 5, §XII, Ex. 19(iii)]:

$$\{\theta,t\} = \{\phi,t\} + \{\theta,\phi\}\cdot\dot{\phi}^2 \tag{2.3.13}$$

where $\{\theta,\phi\}$ is the Schwarzian derivative of the phase θ with respect to the phase ϕ. If this derivative is null, the two phases are connected by a homographic relation [ibidem, Ex. 19(v)], *i.e.*:

$$\{q,f\} = 0 \quad \backslash \quad q(f) = \frac{af+b}{gf+d} \tag{2.3.14}$$

so that we have:

$$\{q,t\} = \{f,t\} \tag{2.3.15}$$

telling us that the homographic action of the 2 × 2 matrices can cover the whole ensemble of solutions of the equation (2.3.12). According to this theorem, the general form of the solution of equation (2.3.12) depends on three parameters: it can be obtained from any particular solution by the group formula (2.3.14). In other words, we can construct the whole system of phases of a signal having a definite frequency, starting from a particular one: the system of phases corresponding to the same frequency – this one being defined by the equation (2.3.12), with an amplitude as in equation (2.3.10) – is the orbit through a particular phase θ of the group of real homographies. This is a continuous group with three infinitesimal generators, locally described as a **sl**(2, R) Riemannian space. This Riemannian space *is the local expression of the holographic phenomenon*, which here has a precise meaning: the whole system of phases corresponding to the same frequency. This gives us a possibility of interpretation – and speaking of interpretation here, we mean interpretation in the wave-mechanical sense, whereby the phase can be associated to a particle [11] – mathematically describable in the terms that follow.

The hard part of the mathematical description of the holographic phenomenon according to the previous definition would be to find the 'seed' phase, the phase whose information is carried in any other phase. Let us assume that we have found it, and denote it by θ. The whole sistem of phases ϕ carrying this information is described by equation (2.3.14), with θ = *constant*. Thus, any phase ϕ is mathematically describable by the solutions of a differential equation of Riccati type:

$$df = w^1 f^2 + w^2 f + w^3 \tag{2.3.16}$$

correlating the variation of phase with the variations of the three parameters describing the holography. Here, the differentials (ω^k) are the components of the standard **sl**(2,R) coframe [see [35], equation (4.4.3)]. If we are able to transform this equation into an ordinary differential equation with respect to a certain 'time parameter', then it gives us an expression of the phase rate to be used in equation (2.3.10), in order to define the amplitude.

Now, in most cases *we* have encountered thus far in our study, this is an easy task facilitated by the metrics of the **sl**(2,R)-type: as a rule, these possess three *Killing vectors*, for which the dual rates (ω^k/dt) are constants along their geodesics (see *e.g*. [45], for mathematical details). It is known, indeed – and we shall repeat the procedure here for a typical case of interest, in the due time, when the occasion will call for it – that the differential forms of the **sl**(2,R)-type coframe are projections of the momentum forms generated *via* the metric Lagrangian, along the Killing vectors. Therefore, in such cases, the equation (2.3.16) becomes an ordinary Riccati differential equation along the geodesics:

$$\dot{\phi} = a^1\phi^2 + 2a^2\phi + a^3 \tag{2.3.17}$$

where (a^1, a^2, a^3) are three constants characterizing the **sl**(2,R)-type geodesics in question, and a dot over means differentiation with respect to the arclength of the geodesics. This means that a geodesic becomes a point in the **sl**(2,R)-type Riemannian space. So, according to the holographic principle, only along such geodesics the physical theory may happen to be interpretable in the wave-mechanical sense. Of course, the process asks for an inversion of the amplitude defined by the rate of phase (2.3.17), so that the inverse of the amplitude will appear as describing a free particle. Indeed, using the combination of the Kepler law (2.3.6) with the equation (2.3.17) gives:

$$\frac{r^2}{a^1 f^2 + 2a^2 f + a^3} = const, \quad A^2 = r^{-2} \tag{2.3.18}$$

which represents the radial motion of a free particle, whose kinematics is described in a time provided by the phase ϕ.

Again, we have strong clues to believe that, physically speaking, this should be the case: according to *Wagner's theorem* (see [34], Chapter 4, §4.3) this holographic space is the realm of the free particles realizing the oscillators. The most important of these clues is the fact that the holographic definition of the frequency characterizes indeed the nucleus of a planetary atom. This statement seems to us sufficiently proven as a consequence of the classical dynamical problem associated to Kepler problem [36]. However, in the theory of nuclear matter *per se*, this idea comes associated with an idea of *interpretation via* the concept of *collective coordinates* [20]. So, we need to insist on the physical aspect of the problem from the perspective of these two natural philosophical concepts. For this we need first some special geometrical considerations.

2.4. SOME DIFFERENTIAL-GEOMETRIC PREREQUISITES

The mathematical method itself, for carrying out the task of introducing the physics into natural philosophy is, in this specific case, based on some almost trivial statements regarding the foundations of the mathematics necessary in building a differential geometry. These statements emerged by and large apparently unnoticed or, even if noticed, they have not been properly used in their capacity; at least not for physical purposes, anyway. In order to make our statement more obvious, we reproduce here two of these statements, in the form of Élie Cartan's 'algebraical' theorems which are recognized to form the ground of his remarkable mathematical approach to differential geometry involving the so-called *moving frames* (for a clear description of the idea, from the point of view we adopt here, see, for instance, [47], Volume II, Chapter 7). Afterwards, these theorems will be used in a short description of the Cartan's method for the classical case of the differential geometry of surfaces.

The theorems in question are drawn here directly from one of Cartan's courses, published *via* the Russian geometrical school of S. P. Finikov (see [9]; pp. 16 – 17, Theorems 7 and 9). We appropriate them for our purposes here under the name of *Cartan Lemmas 1 and 2*, only in order to be suitably used in making our point as explicit as possible. Here they are:

Lemma 1. Suppose that $s^1, s^2,\ldots, s^p$ is a set of linearly independent exterior 1-forms. Then there exists a convenient symmetric matrix, $\boldsymbol{a}$ say, such that:

$$s^{\alpha} \wedge \phi_{\alpha} = 0 \quad \Leftrightarrow \quad \phi_{\alpha} = a_{\alpha\beta} s^{\beta} \quad with \quad a_{\alpha\beta} = a_{\beta\alpha} \tag{2.4.1}$$

where $\phi_1, \phi_2,\ldots, \phi_p$ is another set of linearly independent 1-forms and a summation over repeated indices is understood.

Lemma 2. Suppose the basic differential elements $du^1, du^2, \ldots, du^n$ are connected by a system of equations:

$$\omega^1 = 0, \quad \omega^2 = 0, \quad \ldots \quad \omega^p = 0 \tag{2.4.2}$$

where ω^α, $\alpha = 1, 2, \ldots, p$ are linearly independent 1-forms. In this case the 2-form f constructed with the differentials du^1, du^2, …, du^n vanishes as a consequence of this system of equations if, and only if f can be written as the sum of exterior products:

$$f = \omega^\alpha \wedge \phi_\alpha \tag{2.4.3}$$

where, again, summation over α is understood, and ϕ_α are p *conveniently chosen* 1-forms.

The first one of these theorems is, by and large, known as *Cartan's Lemma* proper in the specialty literature, being routinely used, so to speak. As to the second one of these theorems, it carries no special name in the literature, being in fact used only occasionally.

What appears to be essential in these lemmas, and is almost always stressed mainly in old treatises of geometry, but apparently forgotten lately – perhaps due only to the exclusive mathematical applications – is the fact that the symmetric matrix $\boldsymbol{a}$ from *Lemma 1*, as well as the 1-forms ϕ_α from *Lemma 2*, are *things external to the geometrical problem at hand*, and, moreover, *can be conveniently chosen*. We take these attributions as meaning that they, can be things geometrical, as originally intended, of course. However, for our purposes they can also be *things physical* as well, *i.e.* things through which *the physics can be naturally introduced into geometrical theory*, or *vice versa*: the geometry can be introduced into physical theory.

A case in point: we need to introduce the physics in the theory of surfaces, in order to make it physical, as Louis de Broglie intended, and thus suitable to serve his idea in constructing the light ray, or a ray in general for that matter. When concentrating on the local geometry around a position on a certain surface, without being interested in the global aspects of that surface, as it is almost always the case in physics, especially in the de Broglie's physical optics, this observation becomes essential. Consequently, we can use the above two lemmas, primarily in order to implement physical properties compatible with the geometrical ones or geometrical properties compatible with physical ones. We need to mention, though, that there are a great many problems that the differential geometry allows us to solve using them. In fact, we simply can state that the wave mechanics *per se* would not be possible without these mathematical possibilities.

With this task in mind, and before continuing any further, let us recall once again the convention referring to our use of numerical indices: insofar as either the space or the matter, contemplated as environments in the embedding problem necessary to physical interpretation, are apparently always three-dimensional, we reserve the Latin indices exclusively for this case. The Greek indices are used for any other dimension, as in the case of lemmas above, but especially for dimension two, in the case of surfaces, and dimension four in the case of the manifold of events, *viz*. the spacetime.

One of the most instructive examples of using the calculus with exterior differential forms, of which we shall have to avail plentifully in the present work, is the differential theory of surfaces. The Louis de Broglie's example of construction of a *physical ray*, the structure of the *Ampère current elements* and the physical description of *Thomson*'*s tubes* (see [34], Chapters 2, 5 and 6), are all guiding examples showing the points where we need to intervene by 'inserting' physical conditions in a *local theory of surfaces*. In particular, the definition of the local curvature of a surface in space, and of its variation is a consequence of the *Cartan*'*s Lemmas* just presented above. Inasmuch as these mathematical tools allow us to attach physical reasons to the variation of curvature of surfaces, this makes the fact obvious that this geometric concept has always a physical origin, at least partially anyway, and we shall use it explicitly here. In a historically significant note, for instance, the surfaces were first made known by human senses as space limits of material bodies. More than that, it is, again, highly significant that, when the matter was first made unambiguously responsible for the curvature of space [10], only the surfaces were taken into consideration for analogy, not the space itself.

We need, therefore, an appropriate way of describing the local situations in the case of surfaces – this is what counts most in the case of a theory of rays anyway – as well as some mathematical connection between these situations which, for rational explanation, can be turned to physical descriptions. To wit, we first need to focus on the local situation, by performing the analysis in terms of the components of the position vector on a surface [22]. In a Cartanian version of the local geometry of a surface, we use two coordinate lines with parameters (u_1, u_2), and take the unit vector $\hat{\boldsymbol{e}}_1$ of the reference frame on surface along the lines of coordinates $u_2 = constant$, and the unit vector $\hat{\boldsymbol{e}}_2$ along the coordinate lines $u_1 = constant$, while $\hat{\boldsymbol{e}}_3$ is the local normal to surface. Occasionally, this normal direction is also denoted by $\hat{\boldsymbol{a}}$, the letter suggesting an oriented *area*, therefore a limited portion of surface. Obviously, $u_{1,2}$ denote the parameters on the surface, as we said, but this requirement involves physically special precautions, as we shall soon encounter. Assume that $(\hat{\boldsymbol{e}}_1,\hat{\boldsymbol{e}}_2)$ is a reference frame on the surface: any physical vector referred to the surface can be written in the form:

$$\boldsymbol{V} \equiv V^k \hat{\boldsymbol{e}}_k = V^\alpha \hat{\boldsymbol{e}}_\alpha + V^3 \hat{\boldsymbol{e}}_3 \qquad (2.4.4)$$

where we apply the rule, already announced above, about indices. In the case here the Greek indices take the two values *1* and *2*, for they are referring to a two-dimensional space form: the surface. Four values of indices are reserved for relativity, while five values are usually considered for Kaluza-Klein type physical theories, for instance.

Let us present now the way of writing the *absolute* derivative of a vector, defined as a vector field on the surface. This derivative is itself a vector, as a rule. The differentia 'absolute', added to the presentation of the concept of derivative, is taken here in the sense that it contains both the intrinsic variation, of the given vector – which, by its definition needs to be physical – and its variation due to the fact that it is connected to surface, and the surface itself changes its profile in the neighborhood of any one of its points. This is, of course the notorious case of the propagation of a wave. In order to reveal the necessity and significance of any one of these components of the variation of a vector, let us take notice that, using equation (2.4.4) we can write, using in a first instance the usual rules of differentiation (*viz*. the 'Newtonian' rules):

$$d\boldsymbol{V} = (dV^k)\hat{\boldsymbol{e}}_k + V^k(d\hat{\boldsymbol{e}}_k) \tag{2.4.5}$$

The components of the vector in the left-hand side of this equation are not simply the differentials of the components of $\boldsymbol{V}$, insofar as these last ones need to be 'updated', so to speak, with the contribution of the variation of the reference frame itself. This reference frame varies on the surface in concordance with its *local* topography and, being involved in physical problems connected with interpretation concept, even in concordance with the *instantaneous* topography.

We shall denote the reference frame in a point on surface by the symbol $|\hat{\boldsymbol{e}}\rangle$, as in equation (2.4.5). When discussing the intrinsic geometry of the surface, this 'ket' has only two components – the two unit vectors of the reference frame on the surface – but in general we need to maintain three components in order to account for the connection of the surface with the ambient space. The gist of this approach rests upon the simple observation that the concept of surface does not come to our intellect but only mediated through the existence of the matter in space. So, equations:

$$\left|d\hat{\boldsymbol{e}}\right\rangle = \boldsymbol{\Omega}\cdot\left|\hat{\boldsymbol{e}}\right\rangle \quad \therefore \quad \begin{aligned} \left|d\hat{\boldsymbol{e}}_\alpha\right\rangle &= \Omega_\alpha^\beta \cdot \left|\hat{\boldsymbol{e}}_\beta\right\rangle + \Omega_\alpha^3 \cdot \hat{\boldsymbol{e}}_3 \\ d\hat{\boldsymbol{e}}_3 &= \Omega_3^\beta \cdot \left|\hat{\boldsymbol{e}}_\beta\right\rangle \end{aligned} \tag{2.4.6}$$

are taken to mean that the Frenet-Serret equations on surface are in connection with the 3×3 matrix $\boldsymbol{\Omega}$ describing the variation of the reference frame in space. In (2.4.6) we have used the *orthonormality* of the reference frame in order to split the matrix $\boldsymbol{\Omega}$. This equation further suggests the way to introduce the idea that the reference frame needs to be considered according to its physical origin, a fact which will be properly explained as we go on with our presentation.

The equation (2.4.6) summarizes the system of Frenet-Serret equations describing a moving frame *on the surface*. This system is quite sufficient for writing down the equation (2.4.5) explicitly, in the details needed for the variation of a vector. The feasibility of this construction is assured by the fact that in an Euclidean geometry the reference frame in a point in space is only defined up to an arbitrary rotation, which, as always, secures its possibility as a choice. Thus, in the presence of surface, the three-dimensional equations from (2.4.5) can be written in the following form that accounts explicitly for the existence of the surface:

$$d\boldsymbol{V} = (dV^{\alpha} + V^{k}\Omega_{k}^{\alpha})\hat{\boldsymbol{e}}_{\alpha} + (dV^{3} + V^{\beta}\Omega_{\beta}^{3})\hat{\boldsymbol{e}}_{3} \equiv v^{\alpha}\hat{\boldsymbol{e}}_{\alpha} + v^{3}\hat{\boldsymbol{e}}_{3} \tag{2.4.7}$$

where our convention for indices was used. One can see that, even if the vector $\boldsymbol{V}$ would not have itself a component normal to surface, its differential, which is the one usually called absolute, has such a component due to the participation of the surface itself. It is important to take notice that the differential component in question is the same even in cases where the vector itself has a *constant* normal component to surface: in such a case only the intrinsic components of the differential are changing. Let us get into the details of the very differentials, but this time using the point of view of the surface in describing the space.

The Cartan's approach to geometry is particularly appealing to physics by the fact that we can make the surface meaningful in the definition of the local geometry of the space. This means an explicit construction of the matrix $\boldsymbol{\Omega}$ with its relationship to the metric of surface. Start with the observation that the differential vector $d\boldsymbol{r} \equiv s^{k}|\hat{\boldsymbol{e}}_{k}\rangle$. The components of this vector are differentials, and so are the components of the variations of the reference frame $|d\hat{\boldsymbol{e}}_{k}\rangle$. Exterior differentiation and use of Frene-Serret equations provide the *compatibility* and *structural equations*, that can be symbolically written as:

$$\langle d \wedge s| + \langle s| \wedge \boldsymbol{\Omega} = \langle 0| \quad \text{and} \quad d \wedge \boldsymbol{\Omega} + \boldsymbol{\Omega} \wedge \boldsymbol{\Omega} = \boldsymbol{0} \tag{2.4.8}$$

These equations are basic equations for the local geometry of space. If, in order to get them, we only need to adapt the space reference frame to the surface, then the matrix $\boldsymbol{\Omega}$ for surface may be simply identified with a 2×2 submatrix of matrix $\boldsymbol{\Omega}$ for space. Therefore, the best instructive case would be the one in which the space reference frame is orthonormal itself: starting from this case, it will be more obvious where to go on with the physics, when guided only by the rules of calculus. In an extended form, the equation (2.4.8) provides the system:

$$\begin{aligned} d \wedge s^{1} + \Omega_{2}^{1} \wedge s^{2} + \Omega_{3}^{1} \wedge s^{3} &= 0, \\ d \wedge s^{2} + \Omega_{1}^{2} \wedge s^{1} + \Omega_{3}^{2} \wedge s^{3} &= 0, \\ d \wedge s^{3} + \Omega_{1}^{3} \wedge s^{1} + \Omega_{2}^{3} \wedge s^{2} &= 0 \end{aligned}$$

Now, assume that ***dr*** is an intrinsic surface vector. This occurrence can be expressed as the vanishing of its normal component s^3, which brings this system to the form that should be valid on the surface:

$$d\wedge s^1+\Omega^1_2\wedge s^2=0 \quad d\wedge s^2+\Omega^2_1\wedge s^1=0, \quad \Omega^3_1\wedge s^1+\Omega^3_2\wedge s^2=0 \qquad (2.4.9)$$

Thus, as we said, we take a first instance of the matrix $\boldsymbol{\Omega}$ characteristic to surface, as the 2×2 submatrix of the matrix $\boldsymbol{\Omega}$ for space:

$$(\Omega_s)^\beta_\alpha \stackrel{def}{=} \Omega^\beta_\alpha \qquad (2.4.10)$$

with a proper identification of this submatrix and of the other entries as functions of the position on surface. Actually, this would be more profitable for the construction of a geometry of the ambient space given the surface as reference, than the other way around.

Now, we start using the *Cartan's Lemmas*, as presented in the beginning of this section. Start with the last one of the equations (2.4.9): according to *Cartan's Lemma 1*, the entries $\Omega^3{}_1$ and $\Omega^3{}_2$, of the matrix $\boldsymbol{\Omega}$, should be the components of a ket vector $|\Omega^3\rangle$, that can be expressed *linearly* in terms of s^1 and s^2, *by a homogeneous relation, realized via a conveniently chosen symmetric matrix*:

$$\Omega^3_\alpha=b_{\alpha\beta}s^\beta, \quad b_{\alpha\beta}=b_{\beta\alpha} \quad \leftrightarrow \quad \left|\Omega^3\right\rangle=\boldsymbol{b}\cdot\left|s\right\rangle, \quad \boldsymbol{b}=\boldsymbol{b}^T \qquad (2.4.11)$$

where the upper index "*T*" stands for "transposed".

The symmetric matrix ***b*** is the matrix of the quadratic form commonly known as the *second fundamental form of surface* in the classical theory of surfaces. This quadratic form describes the local shape of the surface, measuring its "departure from flatness", so to speak, therefore, implicitly, the "curvature" of the surface in any direction. In order to make this obvious and, alongside, to exemplify once again the essential distinction between the usual differentiation and exterior differentiation, we shall express now *the second symmetric differential* of the position vector on the surface. Physically speaking, this is the differential that matters: when referred to an appropriate time as a continuity parameter of a motion, arranged, say, with a clock in a convenient order, it offers the acceleration field that plays the part of field intensity of the forces responsible for the physics of a problem involving the presence of a surface. Cases in point: from practical point of view, *the surface of Earth*, and from purely theoretical point of view, *the horizon of black holes*. Historically, the first case helped in characterizing the gravitational field, all the way starting from Galilei and, through Newton, reaching Einstein's relativity that, in turn, created the concept of black hole. The second case, properly developed into a 'membrane paradigm' [41] gave us the possibility to figure out how the matter *per se* may act in generating a reality – accessible either directly or *via* the intercession of some devices – to our senses.

The second differential of the position vector is, in view of (2.4.6):

$$d^2\boldsymbol{r} \equiv ds^\alpha \hat{\boldsymbol{e}}_\alpha + s^\alpha d\hat{\boldsymbol{e}}_\alpha = (ds^\alpha + \Omega^\alpha_\beta s^\beta)\hat{\boldsymbol{e}}_\alpha + (\Omega^3_\beta s^\beta)\hat{\boldsymbol{e}}_3 \tag{2.4.12}$$

and, obviously, it is only a particular case of the equation (2.4.7): this last one is particularly applied to the vector $d\boldsymbol{r}$. The equation (2.4.12) makes it quite obvious that, unlike $d\boldsymbol{r}$, which is an intrinsic vector of surface, the second differential $d^2\boldsymbol{r}$ has also a component normal to surface, as a consequence of the evolution of the reference frame in space. Obviously, this kind of evolution brings in the properties of the surface as components of the differential variation of the vectors. Using the equation (2.4.11) this component is the quadratic form we called before *the second fundamental form* of surface:

$$\hat{\boldsymbol{e}}_3 \cdot d^2\boldsymbol{r} \equiv -d\boldsymbol{r} \cdot d\hat{\boldsymbol{e}}_3 = \Omega^3_\beta s^\beta = b_{\alpha\beta} s^\alpha s^\beta \tag{2.4.13}$$

where the first identity is due to the fact that, by definition, $d\boldsymbol{r}\cdot\hat{\boldsymbol{e}}_3 = 0$, while in the last equality we used the equation (2.4.11). On this occasion it is also worth considering *the first fundamental form* of the surface, which represents the square of the length of $d\boldsymbol{r}$:

$$(ds)^2 \equiv d\boldsymbol{r} \cdot d\boldsymbol{r} = (\hat{\boldsymbol{e}}_\alpha \cdot \hat{\boldsymbol{e}}_\beta) s^\alpha s^\beta \overset{def}{=} h_{\alpha\beta} s^\alpha s^\beta \tag{2.4.14}$$

Here $\boldsymbol{h}$ is the metric tensor of the surface, and, only in order to define its concept, we assumed here the general case of a surface reference frame which, while being 'normal', is however not 'orthonormal' in general. In this case we obviously assume that the third vector of the space frame is perpendicular to the plane $(\hat{\boldsymbol{e}}_1,\hat{\boldsymbol{e}}_2)$. If in reality this is not the case, we can construct easily a vector satisfying the condition as the exterior product $(\hat{\boldsymbol{e}}_1\times\hat{\boldsymbol{e}}_2)$. Using the metric tensor $\boldsymbol{h}$, we can raise and lower the indices of vector components and matrix indices, in case these are tensors, as it happens to be the case with the matrix $\boldsymbol{b}$. Thus, we have, denoting, as commonly done, by upper indices the contravariant components of the vectors and tensors, a 'mixed' matrix $\boldsymbol{b}$, of entries:

$$b^\beta_\alpha = h^{\beta\nu} b_{\nu\alpha} \tag{2.4.15}$$

Using this equation, we define the *mean curvature* and the *Gaussian curvature* of the surface:

$$2H \overset{def}{=} b^{\alpha}_{\alpha}, \quad K \overset{def}{=} \det(b^{\beta}_{\alpha}) \tag{2.4.16}$$

Now, using the fact that the contravariant metric tensor is simply the inverse matrix of the covariant one, we can write the two measures of curvature in the form they are usually taken in the classical differential geometry of surfaces:

$$2H = \frac{h_{11}b_{22} + h_{22}b_{11} - 2h_{12}b_{12}}{h_{11}h_{22} - h_{12}^2}, \quad K = \frac{b_{11}b_{22} - b_{12}^2}{h_{11}h_{22} - h_{12}^2} \tag{2.4.17}$$

When the reference frame on the surface is also ortogonal, the metric tensor in equation (2.4.14) is the identity matrix up to a factor, and the relations from equation (2.4.17) become simpler:

$$2H = b_{11} + b_{22} \equiv \mathrm{tr}(\boldsymbol{b}), \quad K = b_{11}b_{22} - b_{12}^2 \equiv \det(\boldsymbol{b}) \tag{2.4.18}$$

It is, however, worth keeping in mind the general case where the metric of surface is a general quadratic form, not an Euclidean one. First of all, by itself, this does not upset the general conclusions above: because the metric tensor is a symmetric matrix, we can always construct a local orthonormal reference frame by its eigendirections. The discussion above runs in exactly the same way, with s^1 and s^2 the two components of the metric written as a sum of squares. Secondly, there are distinct advantages of the general approach, when the geometry starts being complicated by issues of physics, which is, in fact, our task here. To start with just an observation to be later on elaborated into a full-fledged theory, notice that the second of the equations (2.4.17) roughly offers the ratio between two areas: the 'deformed' local area, with the 'deformation' due to curvature, and the 'global' area, as represented locally by the determinant of the metric tensor of the surface. This property can further offer a general gauging possibility, valid for any dimension, using the concept of curvature [44].

While we are on this subject, let us take notice of the fact that there is also, within the framework of the very same Euclidean theory, *a third fundamental form* of the surface, defined by the square of the matrix $\boldsymbol{b}$. To wit, $d\hat{\boldsymbol{e}}_3$ is an intrinsic vector to surface, just like $d\boldsymbol{r}$ itself and, according to equation (2.4.6) its square is, again, a quadratic form. In the Euclidean differential geometry of surfaces this is commonly termed as the third fundamental form of the surface:

$$d\hat{\boldsymbol{e}}_3 \cdot d\hat{\boldsymbol{e}}_3 = (\boldsymbol{b}^2)_{\alpha\beta} s^\alpha s^\beta \tag{2.4.19}$$

Mention should be made that there are cases where this quadratic form must be taken as a metric of surface when constructing the geometry of space with reference to that surface [42].

3. THE CATEGORY OF MATTER

According to the definition from §2.3, the local holographic property of a medium entertaining waves is decided by its fundamental physical structure: *the resonator*. On the other hand, concurring with its definition by Planck, this fundamental physical structure can only be physically accomplished in a special optical medium: *the Maxwell fish-eye*. Indeed, the Riemannian geometry of the optical paths – apparently, the only mathematics that serves our intellect in judgments regarding this issue of natural philosophy – indicates that only such a medium can accommodate dipoles internally connected by geodesics that are light rays. And as we are used – since Einstein's times leastways – to think that the geodesics are associated with free particles, a first instance of the free particle in the case of a dipole would be the light coming with the optical device represented by the Maxwell fish-eye. Of course, this kind of light needs an apropriate interpretation: it cannot be always that electromagnetic light suggested by the initial Planck's definition of the resonator. As we have seen thus far, a resonator in the case of quantization of matter – a Procopiu resonator – must be realized, according to Katz natural philosophy of charges, by a magnetic dipole: a piece of matter ending in magnetic charges.

Accepting the concept of interpretation, a holographic property of such an optical medium can be expressed by the homographic action of 2×2 matrices, over the one-dimensional system of phases. These phases can be arbitrary functions of coordinates in space: the linear phases of the classical plane waves are just particular cases. The homographic action of the matrices is, in our opinion, expressly required by the very Dennis Gabor's definition of the holographic property, as a condition of coherence of the phases, which has as a first consequence the fixed value of frequency. As we said, the definition can be applied to a general concept of phase, not just to the one connected with the classical idea of plane waves. As we shall show in this chapter, the general idea regarding the space dependence of phases is *via harmonic mappings*. The holography offers possibility of interpretation, with a rational definition of general free particles.

3.1. THE CHARGE AND THE SURFACE DEFORMATION

The definition of a physical surface is made possible by the concept of curvature (see §2.4 above), according to the physical idea of capillary action advanced by Louis de Broglie. Namely, in the case of a capillary tube, an already existing surface is *deformed* on a portion of surface inside the tube: that portion is the surface of a fluid. Thus, the first order of things from the point of view of a mathematics serving the physics here should be the description of such a deformation. The Cartanian approach delineated in §2.4 allows us a concept of deformation that fits the physical-theoretical needs the best possible way: the *infinitesimal deformation* (see [22], §10-4). Accordingly, we define the two surfaces of a de Broglie region we have called 'strange' (see [34], §2.1), limiting this region longitudinally, as two surfaces having a common infinitesimally deformed surface that can play the part of Lorentz neutral surface for this element. Let us start with the mathematical presentation of the idea.

Consider the surface as a *horizon* of the kind serving to make the case for 'membrane paradigm' in the matters of blackholes, but take the matters in the inverse order: we need to describe an infinitesimal deformation of it, destined to serve in the introduction of physics in the manner of introduction of the electromagnetic fields in the membrane paradigm [40,41]. What one further needs is the construction of a function $\boldsymbol{z}(u,v)$, describing the deformation according to equation:

$$\boldsymbol{r}(\varepsilon) = \boldsymbol{x} + \varepsilon \boldsymbol{z} \tag{3.1.1}$$

where $\boldsymbol{x}$ is the position on a reference surface that goes by deformation into position $\boldsymbol{r}$ of the deformed surface. For the construction of $\boldsymbol{z}$, we use the metric form of the surface. In this case, the deformation is infinitesimal if:

$$\frac{ds^2(\boldsymbol{r}, d\boldsymbol{r}) - ds^2(\boldsymbol{x}, d\boldsymbol{x})}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} 0 \tag{3.1.2}$$

where ε is a parameter. According to (3.1.1), we can write the deformed metric as:

$$ds^2(\boldsymbol{r}, d\boldsymbol{r}) = ds^2(\boldsymbol{x}, d\boldsymbol{x}) + 2\varepsilon(d\boldsymbol{x} \cdot d\boldsymbol{z}) + \varepsilon^2(d\boldsymbol{z} \cdot d\boldsymbol{z}) \tag{3.1.3}$$

and then the deformation is infinitesimal in the sense of (3.1.2) if:

$$d\boldsymbol{x} \cdot d\boldsymbol{z} = 0 \tag{3.1.4}$$

In other words, in the first order of the parameter ε, the infinitesimal deformation is not even "felt" in the surface by its metric: what we need, in order to 'feel' it, is a finer perception, reaching into the second order in the parameter ε. Assuming an Euclidean mentality, there is always an arbitrary vector $\boldsymbol{q}$, serving in writing $d\boldsymbol{z}$ in the form:

$$d\boldsymbol{z} = \boldsymbol{q} \times d\boldsymbol{x} \tag{3.1.5}$$

which satisfies the condition (3.1.4) just naturally. The arbitrariness of $\boldsymbol{q}$ may be significantly reduced – and we shall explain what "significantly" means right away – if we take notice that $d\boldsymbol{z}$ has to be an exact differential, for then we must have:

$$d \wedge d\boldsymbol{z} = \boldsymbol{0} \quad \rightarrow \quad d\boldsymbol{q} \times^{\wedge} d\boldsymbol{x} = \boldsymbol{0} \tag{3.1.6}$$

Here "×^" means that in the monomials of the vector product, the usual multiplication needs to be replaced by an exterior multiplication of differentials. Using the notation:

$$d\boldsymbol{q} \overset{def}{=} j^k \hat{\boldsymbol{e}}_k \tag{3.1.7}$$

the condition (3.1.6) can be transcribed in the form:

$$(-j^3 \wedge s^2)\hat{\boldsymbol{e}}_1 + (j^3 \wedge s^1)\hat{\boldsymbol{e}}_2 + (j^1 \wedge s^2 - j^2 \wedge s^1)\hat{\boldsymbol{e}}_3 = \boldsymbol{0}$$

which, in turn, comes down to the system of equations:

$$j^3 \wedge s^1 = j^3 \wedge s^2 = 0, \quad j^1 \wedge s^2 - j^2 \wedge s^1 = 0 \tag{3.1.8}$$

The first two of these equations show, according to the rules of the exterior differential calculus, that $j^3 = 0$ on the surface, because s^1 and s^2 are independent exterior differentials of the first degree in the geometry of a surface described by them. According to the definition equation (3.1.7), this means that the vector $d\boldsymbol{q}$ is situated in the tangent plane of the surface, *i.e.* it can be taken as *an intrinsic vector* with respect to surface, just like $d\boldsymbol{x}$ or $d\boldsymbol{r}$. On the other hand, the last equation from (3.1.8) says something more. First, by the Cartan's *Lemma 1*, it can be transliterated into a matrix equation:

$$\begin{pmatrix} -j^2 \\ j^1 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} s^1 \\ s^2 \end{pmatrix}$$

that is (3.1.9)

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} j^1 \\ j^2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} s^1 \\ s^2 \end{pmatrix}$$

According to its 'intrinsic' property, the vector $\boldsymbol{dq}$ looks like a sort of 'complement' of the infinitesimal displacements $\boldsymbol{dx}$ on the surface, as described in the previous §2.4. The similarity goes even deeper: the conditions for integrability of $\boldsymbol{dq}$ are:

$$d \wedge j^1 + \Omega_2^1 \wedge j^2 = 0, \quad d \wedge j^2 + \Omega_1^2 \wedge j^1 = 0, \quad \Omega_1^3 \wedge j^1 + \Omega_2^3 \wedge j^2 = 0 \tag{3.1.10}$$

and, obviously, replicate the similar ones for the components of $\boldsymbol{dx}$ given in equation (2.4.9). Using the equation (3.1.9), the third one of these conditions amounts to:

$$\alpha c + \gamma a - 2\beta b = 0 \tag{3.1.11}$$

which means that the infinitesimal deformation adds to the support function of the surface a quadratic component, apolar to the second fundamental form. Consequently, this new intrinsic vector $|j\rangle$ offers, in fact, a description of the deformation by an 'update', as it were, of the second fundamental form of the surface. This is, actually, a natural consequence of the surface deformation, in the first place. Still natural should then be an update of the curvature matrix. Let us see how this can be mathematically inferred.

Assuming that the curvature of the surface is essential in its physics, especially in the physics of electricity, we are free to choose to read the third of the equations (3.1.10) as determining the ancillary vector $|j\rangle$ in terms of the curvature, according to the Cartan's *Lemma 1*, so that there is a *convenient symmetric matrix* $\boldsymbol{A}$, such that:

$$\left|j\right\rangle = \boldsymbol{A} \cdot \left|\Omega^3\right\rangle \tag{3.1.12}$$

Now, the previous theory of infinitesimal deformation helps us in establishing a special structure of the matrix $\boldsymbol{A}$, in terms of the variation of curvature. First, when we use the last of equations (3.1.10), in conjunction with the geometrical definition of $|\Omega^3\rangle$ from equation (2.4.11) and with equation (3.1.9), both written formally as:

$$\left|\Omega^3\right\rangle = \boldsymbol{b} \cdot \left|s\right\rangle, \quad \boldsymbol{i} \cdot \left|j\right\rangle = \boldsymbol{a} \cdot \left|s\right\rangle$$

we get from (3.1.12) the following local relation defining the matrix A:

$$\boldsymbol{a} = \boldsymbol{i} \cdot \boldsymbol{A} \cdot \boldsymbol{b} \quad \therefore \quad \boldsymbol{A} = -\boldsymbol{i} \cdot \boldsymbol{a} \cdot \boldsymbol{b}^{-1} \tag{3.1.13}$$

Here, $\boldsymbol{i}$ is the 2×2 fundamental skew-symmetric matrix, as usual: our notation only suggests the obvious fact that it is the matrix replica of the imaginary unit from the case of complex numbers. The relation (3.1.13) is not universally independent on the portion of surface around a certain position. However, it is certainly locally useful, if we are able to detect the possible mechanisms of changing the surface profile.

If the matrix $\boldsymbol{a}$ is determined such that $|\Omega^3\rangle$ is a constant vector – a condition equivalent to the conservation of the normal vector of the portion of surface, which can be taken as a natural definition of a 'portion of surface' necessary in the physics of the de Broglie's ray – then $\boldsymbol{A}$ does not depend, indeed, but only on the existing curvature and its differential variation. This can be seen as follows: according to the definition (2.4.11) of the vector $|\Omega^3\rangle$, its condition of being constant comes down to:

$$d(\boldsymbol{b}\cdot|s\rangle)=|0\rangle \quad \therefore \quad |ds\rangle=-(\boldsymbol{b}^{-1}\cdot d\boldsymbol{b})\cdot|s\rangle \tag{3.1.14}$$

Assuming therefore, the component of vector $|j\rangle$ strictly measured by the variation of curvature, we can take $\boldsymbol{a} = -\boldsymbol{b}^{-1}\cdot d\boldsymbol{b}$, so that equation (3.1.13) can be formally rewritten as:

$$\boldsymbol{A}=\boldsymbol{i}\cdot\boldsymbol{b}^{-1}\cdot d\boldsymbol{b}\cdot\boldsymbol{b}^{-1} \tag{3.1.15}$$

and the equation (3.1.12) becomes:

$$|i\rangle\equiv\delta\boldsymbol{b}\cdot|s\rangle \quad where \quad \delta\boldsymbol{b}\equiv-\begin{pmatrix}\omega^1 & \frac{1}{2}\omega^2+dn\\ \frac{1}{2}\omega^2-dn & \omega^3\end{pmatrix} \tag{3.1.16}$$

where $(\omega^1, \omega^2, \omega^3)$ is the **sl**(2,R) coframe constructed on the matrix $\boldsymbol{b}$, and dn is an exact differential. In other words, by infinitesimal deformation as it is defined in this section, the curvature matrix gathers a differential component also containing a skew-symmetric part in need to be interpreted. Classical-type considerations point out toward a connection of this part with a surface torsion represented by a 'twist' [33].

Consider now the differential 1-form $|\boldsymbol{x},\boldsymbol{q},d\boldsymbol{q}|$ representing the volume of the cuboid constructed on the three vectors entering its expression. Insofar as all three vectors are variable, we may need the variation of this volume. This is also physically significant and needs to be calculated. The exterior differential of it is a 2-form, just like the electric induction or magnetic flux in electrodynamics. This 2-form can be calculated, and we do this job by first rearranging the elementary volume 1-form, such that it appears as:

$$|\boldsymbol{x},\boldsymbol{q},d\boldsymbol{q}|=d\boldsymbol{q}\cdot(\boldsymbol{x}\times\boldsymbol{q})$$

Then its exterior differential is simply the exterior product of two differentials:

$$d \wedge \left| \boldsymbol{x}, \boldsymbol{q}, d\boldsymbol{q} \right| = d(\boldsymbol{x} \times \boldsymbol{q}) \overset{\circ}{\wedge} d\boldsymbol{q} \tag{3.1.17}$$

where the symbol "$\overset{\circ}{\wedge}$" means that in the exterior product the usual multiplication of numbers is replaced with the dot product of the vectors. Now, because, according to the usual rules of differentiation, the first factor here can be written as:

$$d(\boldsymbol{x} \times \boldsymbol{q}) = (q^3 s^2 - x^3 j^2)\hat{\boldsymbol{e}}_1 + (x^3 j^1 - q^3 s^1)\hat{\boldsymbol{e}}_2 + (x^1 j^2 - x^2 j^1 + q^2 s^1 - q^1 s^2)\hat{\boldsymbol{e}}_3$$

one can transcribe (3.1.17) as:

$$d \wedge \left| \boldsymbol{x}, \boldsymbol{q}, d\boldsymbol{q} \right| = q^3(-j^1 \wedge s^2 + j^2 \wedge s^1) + 2x^3(j^1 \wedge j^2)$$

In the right hand, side the first paranthesis is zero by (3.1.11); calculating the remaining term by using the equation (3.1.12), we can finally wrap up the calculation of (3.1.17) in the form:

$$d \wedge \left| \boldsymbol{x}, \boldsymbol{q}, d\boldsymbol{q} \right| = 2x^3(ac - b^2)(s^1 \wedge s^2) \tag{3.1.18}$$

Locally, x^3 is the *support function* of the surface, which can be used as a space coordinate in case we use the local patch as reference.

In order to realize the importance of this definition of infinitesimal deformation, we relate it to the Lorentz's definition of the electric matter. So, we have a charge vector $\boldsymbol{q}$, to be understood as a *triad of numbers* (q^1, q^2, q^3) independently on any reference frame: in short, a ket $|q\rangle$. Likewise, the vector $d\boldsymbol{q}$, effectively characterizing the infinitesimal deformation, has to be understood as a ket $|j\rangle$ representing the intensity of the charge current, having the entries (j^1, j^2, j^3). This current has only in-surface components, for, by the first two conditions (3.1.6) the third component of charge namely $j^3 = 0$, and thus q^3 can be taken as a constant. Apparently, this is the constant alluded to by the definition of Lorentz for the charge. In order to keep this idea fresh, it is worth reproducing, once again, the very words of Lorentz himself. Quoting, therefore:

> If, after arbitrary movements, the matter is reduced to its primitive configuration, and if, during these movements, *every element of a surface which is steadfastly attached to the matter was traversed by equal quantities of electricity in opposite directions, all of the points of system will be found in their primitive positions* ([32], §57; *our translation and emphasis*)

The infinitesimal deformation presented above seems to express mathematically the very essence of this physical point of view. The only new twist we give to this mathematics is that the charge, as a property of matter, becomes the generator of this deformation. We only need to define more precisely what is a surface "steadfastly attached to matter", and afterwards to realize that "traversing" and "primitive positions" have special meaning related to the geometry of motions.

In this connection, another quotation of the great theorist should be clarified based on the geometrical theory of deformation. Once we accept the idea that the infinitesimal deformation is connected to the existence of charge, we need to assume further some difference between a deformed surface and a surface "steadfastly attached to matter". To wit, such a surface cannot be taken as a support of transporting charge: there is a difference between the surface transporting the matter *per se* – the matter "steadfastly attached to surface", as it were – and the surface *containing electricity*. There is a difference between the two fluids – electric and inertial – to be recognized in the fact that the static forces generated by charges prevail over those generated by gravitation. But Lorentz takes the ordinary inertial fluid as a model for any fluid. Quoting:

> If this *hypothesis cannot be admitted in the case of an ordinary fluid*, *it could not be applied to the electric fluid either*. However, this fact does not prevent our equations of motion from being accurate. Indeed, *the mass of this last fluid was supposed to be negligible*, and in calculating the variation δT (*kinetic energy*, *n/a*) only that kinetic energy was considered, *which is specific to the electromagnetic movements*; it will suffice therefore that *the material points liable of these motions*, and which *are not to be confused with the electricity itself*, *enjoy the property of returning to the same positions if, for each surface element, the algebraic sum of the quantities of electricity by which it has been crossed, is 0*.
>
> Now, one is entirely free to try on the mechanism that produces the electromagnetic phenomena *any convenient assumption*, and while recognizing the difficulty of imagining a mechanism that possesses the desired property, it seems to me that we do not have the right to deny its possibility. ([32], §67; *our translation and Italics*)

In order to clearly delineate the physical concepts here, we follow the idea of infinitesimal deformation closely. And this idea takes a positive turn, based on classical considerations: there should be an electrically neutral deformed surface, for any pair of surfaces 'steadfastly attached to matter'.

Indeed, in matters physical, the Ampère current element connected to electricity (see [34], Chapter 5, *passim*) should not be a line element as, usually considered in the classical electrodynamics. Closer to a reality, it should rather be a cylindrical portion of matter included in a portion of tube, in order to support the different physical requirements. The first of these, and foremost we should say, is the finiteness of a physical conductor: the way it apears to our senses, it is *not a line*, but *a solid*. Then the idea of de Broglie's ray comes just naturally when describing such a conductor: the classical current element is simply a matter tube, inside which the electricity propagates. Leaving aside, for the moment, the side cylindrical portion of surface delimiting the element like a ring, the classical theory of surfaces allows us, *via* the concept of infinitesimal deformation, to characterize the Ampère element thus conceived, longitudinally, as it were. And, when we say longitudinally, we mean in the direction of the currents passing through it. It is the direction along which the element extends infinitesimally, just like its classical counterpart. However, unlike that classical ancestor, this Ampère current element has sideways extension that may be characterized by a measure – either finite or infinitesimal – just like the longitudinal extension, which, however, is exclusively infinitesimal.

3.2. A PROCOPIU RESONATOR

Now, we can go for a proper update of the definition of an Ampère element. It seems to be the best candidate for what one can think as a Procopiu resonator: a piece of matter limited by two magnetic charges. The hard part of the problem here is the definition of the magnetic charge, but the infinitesimal deformation allows us to define it. It is in this spirit, that the theory we just presented above assigns the infinitesimal deformation to the existing charge from an all-pervading sea of charge in the form of neutral vacuum, perceived by matter *via* a portion of a surface locally described by the parameters (α, β, γ) characterizing its curvature. This can be, for instance, a typical portion of the wave surface delimited by a de Broglie tube representing a light ray.

Once again, our use of symbols $\boldsymbol{q}$, $d\boldsymbol{q}$, $j^{1,2,3}$ in the previous mathematics is entirely intentional: $\boldsymbol{q}$ represents a charge vector, as usual, while $|j\rangle$ represents a vector intensity of charge current. The charge and its current help us interpret the infinitesimal deformation in terms of the concept of *charge as a vector*. In other words, if the mathematical condition "there is always a vector $\boldsymbol{q}$ serving in writing the deformation" according to equation (3.1.5), is to be backed physically up, this vector should be existent virtually everywhere in the realm thus described mathematically.

The concept of *vector* charge has entered our intellect ever since the charge was approached in physics from the perspective of quantization in matter (see [12] and the literature cited there). Its vectorial character was almost explicit to Julian Schwinger, but we think it was Daniel Zwanziger who can rightfully be credited with a systematization of the concept as a geometrical one (see [55] and the literature cited there). Fact is that, results of the regularization procedure, especially those regarding the so-called *focal regularization* (see [35], §3.4), encourage us into using the concepts of classical dynamics in explaining the geometry of the space containing the center of force of the classical Kepler problem. Therefore, such results can be used in order to explain older concepts, like the *Lorentz*'*s matter* and the *Thomson*'*s realm* with its mandatory *sideways action of force* ([34], §6.2).

Now, regarding the physical structure of currents of this Ampère current element, we assume an interpretation at our disposal. This allows us to define the two portions of surface delimiting the element longitudinally, by "pegging them out" as it were, with "particles" positions. The essential condition for the possibility of such a definition is that the matter inside the current element should be a *Maxwell fish-eye medium*. The general philosophy is then simple: any point within such a medium is the location of entrance and exit of an ensemble of geodesics. The two portions of surface delimiting an Ampère element are "pegged out" by particles serving for interpretation, located in points from the medium. These are corresponding to one another *via* the geodesics of the medium: there are geodesics *exiting* from one particle located on one of the two surfaces, and *entering* another particle located on the other surface, so that the Lorentz's condition is satisfied for such a "dipole". By itself, the dipole is then a resonator, of either Planck or Procopiu type. According to Katz natural philosophy of charges, this last type of resonator is obtained if particles do not have the possibility to move along the line joining them, the matter being rigid. This condition must be retained in order to properly define a magnetic dipole. For now we are limiting ourselves to a few qualitative observations.

The remaining problem is the construction of the mid-surface, satisfying the Lorentz's condition of neutrality. Let us reproduce here, for convenience, a final part of Lorentz's natural philosophy of defining the electric matter, which appears as the crowning point of a long series of works on electricity in the 19th century. These started with Ampère himself, who defined the classical current element, and included the names of Gauss, Riemann, Maxwell, Betti, Beltrami and many other coryphaei of the classical physics and mathematics (see [34] and the literature cited there). Lorentz's definition, succinctly contained in the excerpts above, seems to cumulate all the opportune conclusions of this line of physical thinking, and we intend to follow it further here in order to characterize the Ampère current element. Thus, quoting again:

> Here is now a system of hypotheses that give the value 0 for this variation (*of the kinetic energy of the system*, *a*/*n*):
>
> a. There are *two systems of particles* participating in electromagnetic motions, systems that will be indicated by the letters N and N′.
>
> b. Any time *a certain particle* pertaining to one of these systems *is to be found in the immediate vicinity of a particle of equal mass pertaining to the other system*.
>
> c. The two systems always *have equal movements inversely oriented* or, stating it more exactly:
>
> If two movements of the same duration start with the same initial positions and do not differ but by the sign of the components of the electric current, and if P and P' are points pertaining to systems N and N′ that coincide in the initial configuration, the point P' will reach, in the second movement, the same final position the point P reaches in the first movement.

> This obviously implies that, at the time of coincidence, the points P and P' have equal and opposite velocities. Indeed, changing the signs (*of the components of current*, *a*/*n*) will reverse the velocity of the point P; but, according to the last hypothesis, this velocity must then become equal to that which the point P' had previously.
>
> Notice again that, in the course of a certain movement, a new particle P' will coincide with a given particle P. Two juxtaposed wheels, having equal and opposed rotations of the same axis, may serve as an example. ([32], §69; *our translation and emphasis*, *a*/*n*)

Thus, with no more ado about it, we see the Lorentz program satisfied along the following lines. First, we assume the matter interpreted by ensembles of Hertz material particles in static equilibrium under the three forces generated by their physical differentiae: gravitational mass and the two charges, electric and magnetic. These particles do not exist freely, they are figments of our imagination, just like any physical invention serving our knowledge: material point, dipole and such. The reality is that such an equilibrium does not even exist in our experience: depending on the space scale one Newtonian force prevails over the others in a perpetual nonequilibrium. At the scale of the universe at large, the gravitation prevails, while in microcosmos the electric and magnetic forces prevail. It is starting from this fact of experience that we are able to infer the theoretical necessity of existence of a *static equilibrium*. Inasmuch as this state of the matter is inexistent in our experience, then, naturally, we need to invent it, in the good habit of the classical natural philosophy, or for any human philosophy, for that matter.

Now, once we have at our disposal an interpretation, we can imagine a surface "marked" by material particles serving for interpretation, and take it as a 'surface steadfastly attached to matter', according to Lorentz's expression. A "portion of this surface" is then primarily defined by the values of the entries of matrix $\boldsymbol{b}$ from equation (2.4.11), representing the curvature properties. An infinitesimal deformation, as in the classical capillary tube, will add to this matrix a contribution as in equation (3.1.16). Let us do some calculations: if the matrix $\boldsymbol{b}$ has the entries α, β, γ, so that it can be written as:

$$\boldsymbol{b} \stackrel{def}{=} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \tag{3.2.1}$$

so that if we assume the condition (3.1.14) in defining the element of surface, we end up with the conclusion that the infinitesimal deformation adds to the curvature matrix the contribution (3.1.16), where:

$$\omega^1 = \frac{\alpha d\beta - \beta d\alpha}{\alpha\gamma - \beta^2}, \quad \omega^2 = \frac{\alpha d\gamma - \gamma d\alpha}{\alpha\gamma - \beta^2}, \quad \omega^3 = \frac{\beta d\gamma - \gamma d\beta}{\alpha\gamma - \beta^2} \tag{3.2.2}$$

and:

$$2n = \ln(\alpha\gamma - \beta^2) \tag{3.2.3}$$

According to equation (3.1.16), the curvature of this surface should be expressed by the matrix:

$$\boldsymbol{b} = \begin{pmatrix} \alpha - \omega^1 & \beta - \tfrac{1}{2}\omega^2 - dn \\ \beta - \tfrac{1}{2}\omega^2 + dn & \gamma - \omega^3 \end{pmatrix} \tag{3.2.4}$$

This surface also corresponds *via* the same process of infinitesimal deformation to a surface steadfastly attached to matter having the curvature parameters:

$$\boldsymbol{b} + d\boldsymbol{b} \stackrel{def}{=} \begin{pmatrix} \alpha + d\alpha & \beta + d\beta \\ \beta + d\beta & \gamma + d\gamma \end{pmatrix} \tag{3.2.5}$$

Thus, if the surface characterized by (3.2.1) and that characterized by (3.2.4) are taken as delimiting an Ampère element, the measure of the longitudinal extension of this element should be somehow connected to the metric properties of the mid-surface characterized by the differential forms (3.2.2). As we see them, these metric properties are of a statistical nature (see [36], Chapter 9, §5), along with the curvatures of the surfaces delimiting the element. What we have to retain for now is that a Procopiu resonator can be seen as a special Ampère element of current.

3.3. PHASES AND CHARGES IN A HOLOGRAPHIC UNIVERSE

The holographic principle presented by us in §2.3 involves the homographic action of the 2×2 matrices on the phases. It can, therefore, be inferred that, as long as a surface is not steadfastly attached to matter, it cannot carry matter by transport, but can carry out just phases. As the phenomenon of holography can be described by the homographic action of the matrices, it is expected that the way to describe physically the transport of phases can be mathematically represented by the elements of the actions of matrices. Let us expound on the meaning of this statement.

Naturally, in this context, our first issue here should be the most complete characterization of the 2×2 matrices by their *homographic* and *linear actions* on phases in general. After all, these are the only possible actions that can be defined for the matrices of this kind. Every such matrix, assumed with real entries (α, β, γ, δ) – the notation is chosen in order to be in line with that of the previous work on whose calculations we intend to rely [see [35], equation (4.4.3)] – has *two of its elements* strictly determined by the three independent ratios of its entries. These are the *fixed phases* of its homographic action satisfying the condition:

$$\phi = \frac{\alpha\phi + \beta}{\gamma\phi + \delta} \quad \therefore \quad \gamma\phi^2 + (\delta - \alpha)\phi - \beta = 0 \tag{3.3.1}$$

There are two fixed phases, *viz*. the roots of the quadratic equation just written down here. Within the idea of a holographic property of an optical medium, as described in §2.3, they play a central part in the theory, so we can assume that knowing them is an essential point of understanding of the phenomenon of holography. Our first task here is finding those homographies strictly determined only by the knowledge of their fixed phases.

A first move is to exploit the relation between roots and coefficients in equation (3.3.1), and thereby construct a matrix whose linear action is determined exclusively by its homographic action. The way this statement can be understood will be obvious as we go on with our construction. If *we know* the fixed phases in the holographic process, ϕ_1 and ϕ_2 say, then by equation (3.3.1) two of the entries of the matrix are known alongside, so that the resulting matrix contains two arbitrary parameters besides the fixed phases:

$$\begin{pmatrix} \delta + \gamma(\phi_1 + \phi_2) & -\gamma \cdot \phi_1\phi_2 \\ \gamma & \delta \end{pmatrix} \equiv \delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \gamma \begin{pmatrix} \phi_1 + \phi_2 & -\phi_1\phi_2 \\ 1 & 0 \end{pmatrix} \tag{3.3.2}$$

In this approach, the only matrix strictly determined by the knowledge of the fixed phases must be of the form:

$$\begin{pmatrix} \phi_1 + \phi_2 & -\phi_1\phi_2 \\ 1 & 0 \end{pmatrix} \tag{3.3.3}$$

up to a multiplicative constant. Denoting this matrix by $\boldsymbol{\Phi}$ – suggesting the idea of 'phases' in the construction of a matrix – the most general matrix (3.3.2) having the two fixed points can be formally written as:

$$\boldsymbol{M} = \delta \cdot \boldsymbol{1} + \gamma \cdot \boldsymbol{\Phi} \tag{3.3.4}$$

In other words, determining a matrix by its homographic action results in a family of two-parameter commuting matrices having the same fixed phases. This is a general property: any two commuting matrices have the same fixed phases; reciprocally, the matrices having the same fixed phases commute with each other. There are notable exceptions from this general rule, but they are not concerning us just yet.

The main point of interest in the holographic phenomenon here is that, in any homographic action of one of the matrices from the family (3.3.4), the two fixed phases, ϕ_1 and ϕ_2, are preserved, and they can represent the essential information constant during this phenomenon. This information is preserved in any phase $\boldsymbol{M}(\phi)$, where $\boldsymbol{M}(\phi)$ is the homographic action of the matrix $\boldsymbol{M}$.

Now, regarding the *linear action* of the matrix (3.3.4), it is also characterized by its two fixed elements: *the eigenvalues*. As known, there are two of them, and they reproduce the linear structure of the matrix as given the equation (3.3.4), that is they are linear in the fixed phases, involving the same two arbitrary parameters:

$$m_1 = \delta + \gamma \cdot \phi_1, \quad m_2 = \delta + \gamma \cdot \phi_2 \tag{3.3.5}$$

Notice, now, that the matrix from equation (3.3.3) has eigenvalues ϕ_1 and ϕ_2, so that this particular case can be characterized by a matrix $\boldsymbol{\Phi}$ whose eigenvalues and fixed phases are identical. This is what we understand when we say that the linear action is determined by the homographic action. Of course, the statement can be also taken in the reverse for this case: the homographic action is determined by linear action.

As the measurement results are usually represented by eigenvalues, especially in modern physics, one can see that the results (3.3.5) are not pure measurements: they depend here on the fixed phases. In fact, the reckoning can be very well reversed here, in order to make the two fixed phases dependent on *measurement results*, as it were: the two fixed phases are depending linearly on eigenvalues. This is actually the historical order of things measured in general: an experimental setup must exist in order to execute a measurement. The holographic things just follow this path in a specific way, that is all: record two images of an object in two different phases, and then bring them together into a hologram [19]. A 2×2 family of matrices representing a hologram must reproduce this universal situation revealed to our intellect by Dennis Gabor. And it can be understood, indeed, if we attach to a phase the idea of surface, in the manner of Louis de Broglie (see [34], *passim*).

In order to take this opportunity and insert it into a descriptive physical theory, one needs first to proceed to another construction of the previous matrix, for a case entirely opposite to the previous one. This will consist of the construction of the 2×2 matrix strictly depending on phases, but with the *fixed phases independent on the eigenvalues*. The general idea, in keeping with the historical order, is to separate issues when determining a general matrix. And we have a remarkable case for inspiration, that came with the discovery of spin phenomenon of the electrons. Namely, the idea of spin ½ generated the idea of *isospin*, and we need to go along with this last idea when it comes to introducing the charges. This can be done as follows: build out of two given phases, ϕ_1 and ϕ_2, a matrix having them as fixed phases, but having the eigenvalues ± 1. These two eigenvalues can very well be the two fundamental charges of our world, or the two values of the half-spin of particles, for instance [46]. In any case, such a matrix would have the structure:

$$\boldsymbol{\Phi} \stackrel{def}{=} \frac{2}{\phi_1 - \phi_2} \begin{pmatrix} \dfrac{\phi_1 + \phi_2}{2} & -\phi_1\phi_2 \\ 1 & -\dfrac{\phi_1 + \phi_2}{2} \end{pmatrix} \tag{3.3.6}$$

Like the previous matrix, constructed exclusively on the fixed phases as in equation (3.3.3), this matrix has also the fixed phases ϕ_1 and ϕ_2, indeed, but unlike that version of such a matrix, it has the eigenvalues ± 1. Then, a linear combination with arbitrary coefficients, counterpart of that from equation (3.3.4), say of the form:

$$\boldsymbol{M} = \lambda \cdot \boldsymbol{1} + \mu \cdot \boldsymbol{\Phi} \tag{3.3.7}$$

with the matrix $\boldsymbol{\Phi}$ given in equation (3.3.6), has the eigenvalues ($\lambda \pm \mu$) *independently* on the two fixed phases. However, as in the previous construction, these eigenvalues can also be taken as phases.

Concentrating on this last manner of construction, the matrix $\boldsymbol{\Phi}$ is our channel of introducing the concept of surface in physics. The manner of doing this is simple, and even presents itself quite simply to our intellect: *consider the two fixed phases as coordinates on a surface*. The suggestion comes from the 'classical' – we may be allowed to use this suggestive word here – eigenvalue problem of half-unit spin [46]. In that case, the surface is the *regular unit sphere in space*, and we have:

$$\phi_1 = \cot\frac{\theta}{2} \cdot e^{i\varphi} \quad \phi_2 = -\tan\frac{\theta}{2} \cdot e^{i\varphi} \tag{3.3.8}$$

where θ is the angle of colatitude on the unit sphere, and φ is the angle of longitude. The matrix $\boldsymbol{\Phi}$, constructed according to the recipe from equation (3.3.6), is therefore given by the 2×2 table:

$$\boldsymbol{\Phi} = \begin{pmatrix} \cos\theta & \sin\theta \cdot \mathrm{e}^{i\varphi} \\ \sin\theta \cdot \mathrm{e}^{-i\varphi} & -\cos\theta \end{pmatrix} \tag{3.3.9}$$

and has the eigenvalues ± 1, indeed, as can be easily verified. However, we have a potential situation here, if we continue to see the values of spin in the eigenvalues of a matrix like (3.3.7) constructed with the matrix from (3.3.9). To wit, any other spin should then be represented by the eigenvalues of the corresponding matrix $\boldsymbol{M}$. But these eigenvalues are *a priori* arbitrary, with no selection rules to apply: if such rules exist, they must come out from other considerations.

There is, however, an area of application of this scheme of holography, where this arbitrariness may prove beneficial: *the case of a resonator*. Regarding this as a case of charge measurement, it means the measurement of two charges equal in value and opposite as sign: $\pm e$, where e is the quantum of electric charge, or $\pm g$, where g is the quantum of magnetic charge. These two eigenvalues are independent on the phases of the two kinds of charges, just like in the case of half-spin measurements. Any other case of resonator must come out with the eigenvalues of the corresponding general matrix $\boldsymbol{M}$, viz. $(\lambda \pm \mu \cdot e)$ or $(\lambda \pm \mu \cdot g)$, independently on the two phases of the charges. Notice, however, that these 'measurements' provide *a priori arbitrary charges*, either magnetic or electric, and that the two phases switch their roles in the matrices (3.3.9) describing the two measurements.

Regarding the arbitrariness of the results of charge measurements, the theoretical physics invented the idea of *confinement* in order to appease the things in this case: the elementary particles carrying such charges are confined into physical structures, so that they are not natural in the finite universe of our experience, and cannot be observed in a free state. In a holographic universe, though, this fact appears as quite natural: the charges are only phases to be associated to different surfaces in matter by, say, the Lorentz's procedure [see (Lorentz, 1892); §§57 and 67; see also the previous §§3.1. and 3.2. for details], and they are manifested, indeed, in a universe defined as such by the matrix $\boldsymbol{M}$: the quantization procedure is therefore to be defined here accordingly. Such a universe presents itself naturally, where it was indeed signaled for the first time: it is the nucleus of the planetary model, described as a dynamical Kepler problem [36]. The rest of this work is dedicated to characterizing this realm as a holographic universe: optically speaking, it is *a Maxwell fish-eye medium*.

3.4. THE RESONATOR WITHIN THE ATOMIC NUCLEUS

Having at our disposal a matrix like that from equation (3.3.6) the holographic phenomenon can be described as in equation (2.3.16), because the three components of the **sl** (2,R) coframe are readily available. Based on this observation we will develop here a model of the nuclear matter based on the idea of resonator. What we understand by nuclear matter is an optical medium delimited around the center of force in the classical dynamics describing the Kepler motion by the natural condition of existence of closed orbits. This medium is known to be an **sl**(2,R) metric space, with a metric that can be constructed a priori by the methods of absolute geometry, as in the §2.2 above [36]. The fixed phases to be used in construction of the matrix $\boldsymbol{\Phi}$ from equation (3.3.6) are complex, z and z^* say, so that the resulting matrix is:

$$\boldsymbol{\Phi} \overset{def}{=} \frac{2}{z-z^*}\begin{pmatrix} \frac{z-z^*}{2} & -z\cdot z^* \\ 1 & -\frac{z-z^*}{2} \end{pmatrix} \tag{3.4.1}$$

The complex number z has an algebraical form that includes the initial velocity of the motion, so that we can say that the condition of existence of the closed orbits is of a holographic nature, to start with. The second of the two fixed phases is given by its complex conjugate z^*. Let us work in real phases, for the results are more suggestive: the matrix $\boldsymbol{\Phi}$ from equation (3.4.1) is, in real phases, u and v say:

$$\boldsymbol{\Phi} = \frac{1}{i\cdot v}\begin{pmatrix} u & -u^2-v^2 \\ 1 & -u \end{pmatrix}, \quad z \equiv u+iv \tag{3.4.2}$$

where the imaginary unit i is maintained into the picture in order to make the determinant of the matrix -1, as in the cases where the two fixed phases are real, and the two eigenvalues are ± 1; otherwise, these would be $\pm i$. In general, though, we can dispense with the factor i in the theory, with no significant consequences.

The coefficients of the Riccati equation (2.3.16) representing the set of phases associated with a given hologram generated by matrix $\boldsymbol{\Phi}$ from equation (3.4.2), are [see [35], equation (4.4.3)]:

$$\omega^1 = -\frac{du}{v^2}, \quad \omega^2 = 2\frac{udu+vdv}{v^2}, \quad \omega^3 = -\frac{(u^2-v^2)du+2uvdv}{v^2} \tag{3.4.3}$$

The Cayley-Klein metric of this algebra is, up to a sign, the Beltrami-Poincaré classical metric of the unit disk, given by:

$$\omega^1 \cdot \omega^3 - (\omega^2/2)^2 = \frac{(du)^2 + (dv)^2}{v^2} \equiv (ds)^2 \tag{3.4.4}$$

where s, representing the geometrical arclength, is, again, a phase. The geodesics of this metric can be simply calculated by considering it a Lagrangian, and then solving the Euler-Lagrange equations. These are:

$$vu'' - 2u'v' = 0 \quad \text{and} \quad vv'' + (u')^2 - (v')^2 = 0 \tag{3.4.5}$$

where the accent means derivative on s. One can verify right away that the solutions of these equations are given by cycles, which we write in both the parametric form, as well as in the implicit form:

$$\begin{aligned} u &= u_0 + v_0 \tanh s \\ v &= \frac{v_0}{\cosh s} \end{aligned} \quad \therefore \quad (u - u_0)^2 + v^2 = v_0^2 \tag{3.4.6}$$

Along these geodesics, the differential forms (3.4.3) become:

$$\omega^1 = -\frac{1}{v_0} ds, \quad \omega^2 = 2\frac{u_0}{v_0} ds, \quad \omega^3 = -\frac{u_0^2 - v_0^2}{v_0} ds \tag{3.4.7}$$

and along them the equation (2.3.16) is simply an ordinary Riccati equation with constant coefficients. We stop here for now, as we need to discuss the position of equation (3.4.5) and (3.4.7) regarding the connection between the phase and the host space of the matter.

The parameter s can be taken as a phase, provided it satisfies the Laplace equation in space. This can be shown as follows [3] (see also [4] for the description of the method in a particular case): the problem of interpretation in the case of holography is pending on a particle moving on the geodesics (3.4.6). The interpretation *per se* is given by the harmonic mappings from the matter to space. Proceeding as usual [36], we need to construct the energy functional of the metric (3.4.4), which is the integral:

$$\frac{1}{2} \iiint_{Volume} \frac{(\nabla u)^2 + (\nabla v)^2}{v^2} d(volume) \tag{3.4.8}$$

The Euler-Lagrange problem for extremizing this functional, provides the equations:

$$v\nabla^2 u - 2\nabla u \cdot \nabla v = 0 \quad \text{and} \quad v\nabla^2 v + (\nabla u)^2 - (\nabla v)^2 = 0 \tag{3.4.9}$$

These equations look like the equations of the geodesics (3.4.5). The resemblance is becoming an identity if we assume that they represent waves with particles moving *collectively* but along separate geodesics, like the light does along straight lines, in our experience. Indeed, this means that the parameter s of the geodesics needs to be taken as a wave phase: a function in space associated to the particles moving along the geodesics. In this case, the equation (3.4.9) can be written as:

$$vu'' - 2u'v' = -vu'\frac{\nabla^2 s}{(\nabla s)^2}, \quad vv'' + (u')^2 - (v')^2 = -vv'\frac{\nabla^2 s}{(\nabla s)^2} \tag{3.4.10}$$

which reduces to (3.4.5) if s is a solution of the Laplace equation:

$$\nabla^2 s = 0 \tag{3.4.11}$$

The identity in question takes place only for phases that are solutions of the Laplace equation. The equation (3.4.7) associates with one of these geodesics a matrix analogous to the one from equation (3.4.2):

$$\boldsymbol{\Phi} = \frac{1}{v}\begin{pmatrix} u & -u^2 + v^2 \\ 1 & -u \end{pmatrix} \tag{3.4.12}$$

corresponding to the values u_0 and v_0 of the parameters. One can say that a geodesic maintains two phases constant along it: $u \pm v$. These are the fixed phases of the matrix (3.4.12), and they are the same all along a given geodesic. This is, in fact, the method of associating a phase to a moving particle: the free particle moves along a geodesic, and has two real phases constant along that geodesic. But the whole advantage of associating phases to motion this way, comes with another, by far more important observation.

If a geodesic (3.4.6) is uniquely characterized by a matrix like (3.4.12), this means that the correspondence between two geodesics can be related to the variation of such a matrix. A rational model of such a variation presents itself *via* the differential of the matrix:

$$\boldsymbol{\Phi} + d\boldsymbol{\Phi} = \boldsymbol{\Phi} \cdot (1 + \boldsymbol{\Phi}^{-1} \cdot d\boldsymbol{\Phi}) \tag{3.4.13}$$

Let us calculate the matrix $\boldsymbol{\Phi}^{-1}\cdot d\boldsymbol{\Phi}$, using equation (3.4.12). The result is:

$$\boldsymbol{\Phi}^{-1}\cdot d\boldsymbol{\Phi}=\begin{pmatrix}\omega^2/2 & -\omega^3\\ \omega^1 & -\omega^2/2\end{pmatrix}\tag{3.4.14}$$

where the entries are the following differential 1-forms:

$$\omega^1=\frac{du}{v^2},\quad \omega^2=2\frac{udu-vdv}{v^2},\quad \omega^3=\frac{(u^2+v^2)du-2uvdv}{v^2}\tag{3.4.15}$$

This is an **sl**(2,R) coframe, seeing its structural equations:

$$\begin{gathered}d\wedge\omega^1+\omega^1\wedge\omega^2=0,\\ d\wedge\omega^2-2\omega^3\wedge\omega^1=0,\quad d\wedge\omega^3+\omega^2\wedge\omega^3=0\end{gathered}\tag{3.4.16}$$

The Killing-Cartan metric of the coframe (3.4.15) is conform-Lorentzian:

$$\mathrm{tr}(\boldsymbol{\Phi}^{-1}\cdot d\boldsymbol{\Phi})^2\equiv 2\det(\boldsymbol{\Phi}^{-1}\cdot d\boldsymbol{\Phi})=2\frac{(du)^2-(dv)^2}{v^2}\tag{3.4.17}$$

The geodesics of this metric are given parametrically as:

$$u(\varphi)=u_0+v_0\tan\varphi,\quad v(\varphi)=\frac{v_0}{\cos\varphi}\tag{3.4.18}$$

where φ is the arclength along this metric. In other words, going along the geodesics of the metric (3.4.17), the geodesics (3.4.6) of the metric (3.4.4) must be located as:

$$(u-u_0-v_0\tan\varphi)^2+v^2=\frac{v_0^2}{\cos^2\varphi}\tag{3.4.19}$$

We have here a family of cycles through two fixed points $u_0\pm v_0$: these two phases represent two locations satisfying the essential requirement of the Maxwell fish-eye optical medium. Thus, they reproduce the structure of a resonator, and it remains to be decided what kind of resonator it is: *Planck's or Procopiu's*!?

The answer that we find the most rational of them all is based on the observation that the metrics (3.4.4) and (3.4.17) satisfy to a kind of 'duality', if we may be allowed to say so. As we have seen thus far, the parameters u_0 and v_0 of the geodesics (3.4.6) follow the Riemannian geometry of the metric (3.4.17). Let us see what geometry follows the parameters u_0 and v_0 of the geodesics of the metric (3.4.17), in their turn. Just as in the first of these cases, we just need to calculate the differential forms (3.4.15) along the geodesics (3.4.18). The result is:

$$w^1 = \frac{1}{v_0}dj\ , \quad w^2 = 2\frac{u_0}{v_0}dj\ , \quad w^3 = \frac{u_0^2 + v_0^2}{v_0}dj \tag{3.4.20}$$

In other words, a geodesic of the metric (3.4.17) is uniquely associated with a matrix (3.4.14) of the form (3.4.2):

$$\boldsymbol{\Phi}^{-1} \cdot d\boldsymbol{\Phi} = \frac{1}{v}\begin{pmatrix} u & -u^2 - v^2 \\ 1 & -u \end{pmatrix} \cdot dj \tag{3.4.21}$$

This time we have a resonator with two complex locations: $u \pm iv$, prone to be associated with charges in the nucleus of the planetary model, judging by the realm to which the metric (3.4.4) applies.

The bottom line is that there are, indeed, two kinds of resonators inside the nuclear matter thus understood, and deciding which is which takes more than a simple mathematical observation. Our thesis is that it takes a constitutive law of the optical medium in question, and this can be decided only by a physical characterization of the other category involved in the equilibrium within a Wien-Lummer enclosure: the light. Up to this point we discussed only the matter: let us, therefore, turn to the light.

4. THE CATEGORY OF LIGHT

The physical theory of light constructed by Augustin Fresnel had many connotations of quite different sorts. For once, it meant a severance of the theory of light from the old classical phenomenology involving only the phenomena of reflection and refraction. For, by concentrating on the diffraction phenomenon, as it did, Fresnel's theory added, actually, the diffraction to that old phenomenology, thus creating a new phenomenology of light destined to make the theory, depend exclusively on a *technology* of experimentation. In turn, this new phenomenology had to wait for another century or so, in order to be again 'updated', so to speak, with the phenomenon of holography, which can be safely connected with the name of Louis de Broglie, once it is based on quantization [19]. However, the one connotation of Fresnel's theory we are considering now means a generalization of the theory of light, far and beyond its electromagnetic stance, in fact well into the modern theory of particles and fields. In time, this was achieved, first *via* the Yang-Mills generalization of Maxwell electrodynamics and, secondly, *via* the Willem de Sitter's generalization of the concept of free spacetime, setting the Einstein's ideas on relativity in order.

This capability of the Fresnel's theory of being so versatile is due to the fact that it has strong ties with two classical ideas of a general natural philosophical class: the *concept of Ampère element*, as involved in the description of a kind of Lorentz matter and, above all, the *concept of de Broglie*'*s surface*, as the one involved in the classical theory of light. In order to document this statement, we start by quoting the great Henri Poincaré, who once aptly summarized what we think is the essential point of the Fresnel's physical theory of light:

> This is, in a nutshell, the theory of Fresnel. It is in every respect in conformity with the experimental laws; but we notice that it rests upon two hypotheses demanding closer examination. These two hypotheses can be enunciated as:
>
> 1° The elastic force arisen by the motion of a plane wave is independent on the direction of the plane of wave, it depends only on the direction of the vibrations of the molecules, and is proportional to the force developed by an isolated molecule, the other molecules from the plane of the wave remaining at rest.
>
> 2° The only effectual component of the elastic force is the component parallel to the wave plane.
>
> The first of these hypotheses, which Fresnel vainly tried to justify, is entirely arbitrary, but nothing precludes its acceptance (…)
>
> As to the second one, it is an immediate consequence of the incompressibility of the ether. We already stated that, in his calculations, *Fresnel admitted, often implicitly sometimes, that the resistance of the ether to compression would be null, sometimes that it would be infinite*. In these lessons we have situated ourselves, up to this point, in the first of these hypotheses; let us look now for the equations of motion within the hypothesis that the resistance to compression is infinite, that is the elastic medium is incompressible. [(Poincaré, 1889), pp. 229 – 230; *our translation and emphasis*, *n*/*a*]

We used this "second hand", as it were, quotation on the Fresnel's original concepts (Fresnel, 1821, 1827), because it is the clearest one when it comes to the physics involved into description of the light phenomenon. As these few phrases of the illustrious Henri Poincaré suggest themselves, the original expression of the Fresnel's ideas is, inherently we should say, in view of the novelty of the concepts it introduces, a little confusing. This is why we preferred the clarity of this quotation, brought by at least fifty years of "theoretical physics" contemplation of the initial concepts. Notice that Fresnel implicitly considered the *concept of surface* in its utmost generality – once he used the idea of wave plane – which, for once, compels us into construction of an adequate theory, accounting *explicitly* for this concept. The problem of Fresnel inconsistency regarding the description of behavior of the ether to compression, needs also to be undertaken from a modern perspective, for it is connected with a fundamental way of thinking that led in time to the Maxwellian theory of light.

Indeed, it seems to us that the two apparently conflicting physical properties of the ether – sometimes resistance to compression null, sometimes infinite – cannot describe the same physical entity for, according to the common experience, such an entity can possess either one or another of them, but not both properties at once. As one can clearly understand from the above excerpt, the strategy of physics was always to see to what each one of the properties leads in terms of the perceived properties of the ether, and then decide what this medium should really be, in order to check experimentally if true. Useless to say, what the ether actually is has not been decided even to this day, so that, in a way, we are presently giving a reason to this very fact: the two mutually exclusive properties are only a natural philosophical consequence of *the mechanical constitutive characterization of a material continuum*, and as long as the ether is considered material, it cannot be but an *electromagnetic ether*.

4.1. CONSTITUTIVE CHARACTERIZATION OF RESONATOR ENSEMBLES

In the mechanics of continua, one works, as a rule, with *second order tensors* or, more general, 3×3 matrices, in order to represent *stresses* and *deformations* of continua. Unlike the central forces, these are strongly non-polar mathematical things, at least as long as we do not specify them in terms of fields of displacements and forces. Furthermore, when it comes to the reality of these things, it is only guaranteed by the so-called *constitutive law*. Let us elaborate a little on this concept. In broad terms, the mechanical constitutive law is a relationship between stresses and the strains they induce – the measures of deformation – during the process of deformation. As our representations of these concepts is usually by matrices, the most general constitutive law is simply a mathematical relation – algebraic or analytic – between two 3×3 matrices. If we denote by $\boldsymbol{t}$ the matrix of *tensions* (stresses) and by $\boldsymbol{d}$ the matrix of *deformations* (strains), representing, then a constitutive law is a relation of the form $\boldsymbol{t} = \boldsymbol{C}(\boldsymbol{d})$ where the matrix function $\boldsymbol{C}$ is accessible to experimental evaluation. Here we insist on the meaning according to which $\boldsymbol{t}$ is the *applied* stress on a continuum, while $\boldsymbol{d}$ is the *resulting* strain.

The reality we just mentioned above is then connected to the identity of the material characterized by the constitutive law. For, it is claimed, in the modern science of materials, that the stress and strain matrices are universal mathematical tools, while the function $\boldsymbol{C}$ is a *specific feature* of the material upon which the stresses are applied. One can see in the concept of stress, extended beyond the applied stress, a mean to eliminate the force in general from the mechanics of continua. Indeed, it is only the applied stress that is intimately tied up with the idea of force. Otherwise, the stress can be thought of as a *density of energy* describing the matter, in general even independently of the applied force. Therefore, if it is to extend natural philosophical conclusions – which, by their very nature, carry entirely the mark of our senses – onto the description of a fictitious matter, as in the case of ether, for instance, then it is more appropriate to accept the idea that the ether of space deforms in any conditions, and if the ether of matter is acted upon in any way, it responds by the deformation which we describe by a matrix designated $\boldsymbol{d}$. Thus, we are bound to find a function $\boldsymbol{C}$ that implicitly contains the physical nature of this continuum.

Now, a deeper insight into this problem shows a specific feature of it, that goes beyond the physics of central forces: one has to deal here with *uncontrollable manifestations* of the matter. This is perhaps the main deep reason of maintaining the mechanical manner of thinking in physics. It is, indeed, true that every one of our physical actions is associated with forces. In other words, in doing experiments we need control. However, we cannot control but forces and, if anything else, *through* forces. It is seldom noticed that, in the framework of mechanics, as long as we are maintaining the forces as essential theoretical tools, there can be no room for uncontrollable quantities. This is exactly what has happened with the ether theories along the time: nothing uncontrollable has been admitted in its physics. We think that an ether theory is a critical field where we must recognize the *existence of uncontrollable quantities* and, most importantly, we must not describe them in the manner we describe the controllable quantities. If the ether exists, the free motion of the bodies through it is enough proof that we cannot control its deformations.

It just happens that the most general idea of uncontrollability comes quite intuitively with what we would like to call a *natural constitutive law*. Indeed, a constitutive law relating the applied stress to a resulting strain experimentally recorded, usually by a measured length, must be of the form:

$$\boldsymbol{t} = \boldsymbol{p_0} \cdot \boldsymbol{1} + \boldsymbol{p_1} \cdot \boldsymbol{d} + \boldsymbol{p_2} \cdot \boldsymbol{d}^2 \tag{4.1.1}$$

where $\boldsymbol{1}$ is the identity 3×3 matrix. We call this equation **1** a *natural* constitutive law, on the grounds that it can be derived from the very basic considerations on our representations of stresses and strains. Indeed, if our models of stress and strain are 3×3 matrices, and if the constitutive law should be analytic, the equation (4.1.1) must be automatically in effect. For, then, the relation between the two matrices can be represented by a formal series reducible to a second order polynomial *via* Hamilton-Cayley theorem. By the same token, that relation can just as well be written with the places of stress and strain matrices interchanged. Thus, the strain matrix as a function of stress matrix is also a quadratic function, only with some other coefficients.

Now, the deforming *medium* is supposed to have here a precise identity, for we can identify it by the coefficients (p_0, p_1, p_2) which are accessible to experiment, and there are indications that these coefficients are different for different materials. Their values offer what is actually meant by a 'material characterization'. Often times in the actual engineering practice these coefficients are considered *pure material properties,* but this restriction confuses the issues, sometimes with serious consequences, mostly in engineering problems. Let us make this statement a little more precise. No matter what the material properties 'incarnated', as it were, into the coefficients $p_{0,1,2}$ are, we can imagine the following experimental approach to constitutive equation (4.1.1). Consider that in each and every one of the loading experiments involving a stress incurred to a body like the confined ether, a system of tensions can be defined, the principal directions of which coincide with the principal directions of resulting measurable strain. In this case, if $t_{1,2,3}$ are the principal values of this stress matrix, and $d_{1,2,3}$ those of the strain matrix, according to the constitutive law (4.1.1) we must have satisfied the system of three equations:

$$\begin{aligned} t_1 &= p_0 + p_1 \cdot d_1 + p_2 \cdot d_1^2, \\ t_2 &= p_0 + p_1 \cdot d_2 + p_2 \cdot d_2^2, \\ t_3 &= p_0 + p_1 \cdot d_3 + p_2 \cdot d_3^2, \end{aligned} \tag{4.1.2}$$

This system can be considered as a linear algebraic system in the three parameters (p) describing the very material. Nothing hinders us then, in assuming that we are able to perform such experiments allowing us to *measure all three principal values of strain* corresponding to the three principal values of stress at once. For, to be sure, we can always control a state of applied stress, but can only "watch" the resultant strains, and measure them at most. The outcome of experiments – whereby the stresses are controlled, and the strains are only observed, at the very best measured – will then allow us to calculate the material properties embodied in the coefficients $p_{0,1,2}$ from the system (4.1.1). This system has a nontrivial *unique* solution for these coefficients – a solution of the kind required by the uniqueness of the physical properties to be used in the description of the material upon which the loading experiments are performed – if, and only if, the principal determinant of the system (4.1.2):

$$\begin{vmatrix} 1 & d_1 & d_1^2 \\ 1 & d_2 & d_2^2 \\ 1 & d_3 & d_3^2 \end{vmatrix} \equiv (d_2 - d_3) \cdot (d_3 - d_1) \cdot (d_1 - d_2) \tag{4.1.3}$$

is non-null. Thus, the parameters p_0, p_1, p_2 are uniquely determined, regardless of the character of imposed stress, by the solutions of the system (4.1.2) if, and only if the resulting principal deformations are *all different* from one another. We have nonetheless to notice that in such a situation the stresses defined by (4.1.2) are hardly controlled like in an actual experiment: they are, in fact, totally uncontrolled. Now, coming back to our experimental considerations, no matter how unique, and thereby well suited for characterizing the deforming medium physically, the coefficients thus obtained are by no means *pure material properties*, inasmuch as they all depend on the *impressed state of stress*. Therefore, we are further required to make more precise what we understand by *pure material properties*, and this is, and indeed always was, a big issue.

We can address this issue by noticing that *there are* deformations even in case where there are no impressed stresses acting on our material. The propagation of waves is a case in point, but we do not need to go quite that far, for the gravity makes a very good and comprehensible case: our experience is simply determined in a background dominated by gravity. Based on the discussion above we can further argue that any field must have this essential property. More to the point, as long as we do not know their mechanical origin, such deformations can be considered as some *intrinsic properties* of the deforming medium in question. If it is to extend our experience beyond observations, then such intrinsic properties can be supposed to be generated by forces on whose presence we have momentarily no idea, therefore by those forces assumed to exist inside interpretative ensembles of the matter. Limiting, at least when it comes to the description of ether, the mechanics only to *external* or *impressed* forces, *i.e.* accepting that there is no possibility to describe the action upon continuous parts by forces, leaves no alternative but to consider them as intrinsic properties of the continuum. In terms of system (4.1.2) they can be described by the system of equations:

$$
\begin{aligned}
0 &= p_0 + p_1 \cdot d_1 + p_2 \cdot d_1^2, \\
0 &= p_0 + p_1 \cdot d_2 + p_2 \cdot d_2^2, \\
0 &= p_0 + p_1 \cdot d_3 + p_2 \cdot d_3^2,
\end{aligned}
\tag{4.1.4}
$$

Then, the matter description by experiment is transferred into finding the solutions of this homogeneous linear system, in case they exist. As a matter of fact they always exist, we have to decide just how many of them, and this fact depends on what we can *always* really measure. If we are able to always measure three different deformations in three different directions in space, in the case of no apparent action on the medium, then this medium is not responsive to any impressed stresses ($p_{0,1,2} = 0$). That much we know from our historical experience: the possibility of bodies to move unobstructed through ether, is the main quality of the ether that propagates light and this is why Fresnel 'had sometimes' to assume it!

However, there are also possibilities of solutions in which the ether may be responsive to external stresses, in other words its deformation can be associated with stresses definable according to equation (4.1.4) for the material. These possibilities are given by the nontrivial solutions of the system (4.1.4), and they are made possible only in those cases where there are less than three measured strains. Thus, if we measure one and the same strain value in any direction – the material is isotropic from the point of view of deformation – we can have a double infinity of states of stress of ether, depending on *two matter parameters*. If we measure two strain values, and only two, in a direction and its perpendicular plane for instance, then we have states of stress of the ether depending on *one matter parameter*. Granting that we can include one of the matter parameters into a measurable quantity, the most general constitutive law satisfied by the ether exhibiting definable stresses corresponding to a measured strain will be given by a constitutive law involving three determinable parameters in the form:

$$\boldsymbol{t} = K(\boldsymbol{d} - d_1\boldsymbol{1})(\boldsymbol{d} - d_2\boldsymbol{1}) \tag{4.1.5}$$

where K is an arbitrary constant. One can say that such a material has *three* uncontrollable quantities, out of which *two* are measurable: the constant K and the two eigenvalues of deformations.

In closing here, notice that as long as we are interested in just the measurable quantities in characterizing the ether, there is a convenient way to do this. Namely, we just have to construct a characteristic deformation matrix of the deforming medium, using the two uncontrollable strains measured under no apparent mechanical action upon the medium in order to form a special tensor having the entries:

$$d_{ij} = d_2\delta_{ij} + (d_1 - d_2)\cdot l_i l_j, \quad i,j = 1,2,3 \tag{4.1.6}$$

where $\boldsymbol{l}$ is the unit eigenvector corresponding to the eigenvalue d_1. Such a medium has distinguished directional properties, with respect to the direction $\boldsymbol{l}$, and these properties are given by the eigenvalues d_1 and d_2. As a matter of fact, the equation (4.1.6) does contain both on the previous two cases as particulars, if we agree to characterize the *intrinsic material properties as deformations*. Notice that this is an assumption independent of the constitutive description, and must be secured only by our measurement capabilities. Thus we have this general conclusion: *whenever the medium deforms freely*, *i.e. under no noticeable actions on it*, *its deformation matrix must be of the form given by equation* (4.1.6), *all the particular cases included*. The deformations as well as the stresses are then manifestly tensors. This is the case of the ether in space: the medium through which the bodies move freely.

Notice now that the quadric associated to the tensor (4.1.6) is a spheroid, prolate or oblate, depending on the ratio of the two eigenvalues d_1 and d_2. This is the general geometrical form associated with a dipole. Consequently, we can assume that this kind of ether is simply one of the category supporting the rays described by us in §2.1.

By the same token we can discuss the ether in matter: that category of ether capable of sustaining stresses and exhibiting no strain. It is indeed by this essential property of matter that comes first to our senses in the form of incompressibility. For this, the converse constitutive law must be taken in consideration, namely:

$$d = q_0 \cdot 1 + q_1 \cdot t + q_2 \cdot t^2 \tag{4.1.7}$$

This time, however, ***t*** may be abusively called stress, if we define the stress by controllability: let us just say that it is a tensor representing the internal energy in matter. Then the defining state of such an ether will be characterized by the system of equations:

$$\begin{aligned} 0 &= q_0 + q_1 \cdot t_1 + q_2 \cdot t + q_2 \cdot t_1^2 \\ 0 &= q_0 + q_1 \cdot t_2 + q_2 \cdot t + q_2 \cdot t_2^2 \\ 0 &= q_0 + q_1 \cdot t_3 + q_2 vt + q_2 \cdot t_3^2 \end{aligned} \tag{4.1.8}$$

corresponding to *no strain response*. Again, the characterization of this ether depends upon the number of solutions of this system: if one can always measure three different stresses in three different directions then the ether has no deformational response to any stresses ($q_{0,1,2} = 0$). This is, again, the case "sometimes" considered by Fresnel, *i.e.*, the incompressible case. And the most general strain this kind of ether can exhibit is of the form:

$$\boldsymbol{d} = K_1^{-1} \cdot (\boldsymbol{t} - t_1\boldsymbol{1}) \cdot (\boldsymbol{t} - t_2\boldsymbol{1}) \tag{4.1.9}$$

where the constant K_1 has dimensions of stress. It is perhaps of some gnoseological significance – at least for guiding our natural-philosophical reasoning, if nothing else – that the relation (4.1.9) with $t_1 + t_2 = 0$ has been found to be characteristic for metals, in large as well as small deformations: metals always struck our senses by their *hardness*.

Again, as long as we are interested in just measurable quantities characterizing such a material, then its intrinsic stress tensor assumes a convenient representation, similar to (4.1.6). However, this time we are compelled to assume further that the eigenvalues of ***t***, whatever this physical magnitude may be, are measurable. All we know about them is that they are 'stress-like' as it were, *i.e.*, they must have the physical dimensions of the energy density. Inside a material these can be realized only by interpretation, for instance by fluxes of intermolecular forces, as once posited by Augustin Cauchy. Thus, the counterpart of (4.1.6) is here:

$$t_{ij} = t_2\delta_{ij} + (t_1 - t_2) \cdot m_i m_j, \qquad i, j = 1, 2, 3 \tag{4.1.10}$$

where $\boldsymbol{m}$ is a unit vector corresponding to the eigenvalue t_1. One can say that the general characteristic of materials exhibiting no deformation under stress is of the form (4.1.9), all particular cases included.

A digression is now in order, for better understanding the issues of this characterization of matter. It involves either the constitutive relations (4.1.6) or those from the equation (4.1.10). However, while the first of these descriptions of matter asks only for properties of a continuum – one needs to measure just strains – the second one, involving the equation (4.1.10) asks for more. Namely, as the case of Cauchy stresses shows it, here we need an interpretation of the matter. It pays for later developments to notice that, while in the first case the matter, as a continuum, is described by a Newtonian density, in the second case the description needs an Einsteinian density of numerical type. Interestingly enough, the electromagnetism seems to cumulate the two descriptions of the ether, into the so-called *electromagnetic ether*, as we shall see right away.

The case of equations (4.1.6) and (4.1.10) is specific for matrices that we would like to call as "equivalent" to a vector field: they characterize *dipoles* of two different kinds. We understand this equivalence in the following way: having a vector field $\boldsymbol{v}$, we can construct the following matrix using two parameters α and β:

$$v_{ij} = \alpha\delta_{ij} + \beta v_i v_j \tag{4.1.11}$$

Now, it is clear that, because v_k are the components of a vector, and supposing α and β scalars, gives v_{ij} the components of a *tensor*. One of the eigenvalues of this tensor, namely α, is double. The other eigenvalue, different from α, is given by:

$$\alpha' = \alpha + \beta v^2 \tag{4.1.12}$$

Notice some interesting features of this kind of tensor. First of all, if either β or v_k is null, v_{ij} is a purely spherical tensor. Secondly, if we calculate the eigenvector of the tensor $\boldsymbol{v}$, corresponding to the eigenvalue (4.1.12), we find out that this eigenvector is just the vector $\boldsymbol{v} \equiv |v\rangle$, up to a normalization factor. This property is independent on the parameter α, and this is what we mean by the above mentioned equivalence: given the vector $|v\rangle$ we can directly construct the tensor $\boldsymbol{v}$ as a family of two-parameter tensor matrices having it as an eigenvector. One can say that $\boldsymbol{v}$ represents a kind of action that points in the *general direction* of $|v\rangle$, as it were, not *exactly* in that direction.

One way to get the characterization of fundamental structure of ether, compatible with the category of vacuum – considered as matter, but missing the interpretation – is by admitting that this structure is described not by one tensor of the general type (4.1.11) but by two, with two characteristic vectors, ***u*** and ***v*** say. According to such a logic, the tensor describing the complete fundamental structure of ether would then have entries depending on three parameters:

$$w_{ij} = \alpha\delta_{ij} + \beta u_i u_j + \gamma v_i v_j \tag{4.1.13}$$

This line of ideas is, of course, inspired by the electromagnetic theory of light, where the tensor:

$$w_{ij} = \lambda u_{ij} + \mu v_{ij} \quad \therefore \quad \boldsymbol{w} \overset{def}{=} \lambda\boldsymbol{u} + \mu\boldsymbol{v} \tag{4.1.14}$$

represents the so-called *Maxwell stresses* of the ether. Here λ and μ are some real parameters, describing the "degrees of light and matter" into this ether, with the matrices ***u*** and ***v*** defined as the tensors:

$$u_{ij} \overset{def}{=} u_i u_j - \frac{1}{2}\boldsymbol{u}^2\delta_{ij}, \quad \boldsymbol{u}^2 \equiv \boldsymbol{u}\cdot\boldsymbol{u}, \quad v_{ij} \overset{def}{=} v_i v_j - \frac{1}{2}\boldsymbol{v}^2\delta_{ij}, \quad \boldsymbol{v}^2 \equiv \boldsymbol{v}\cdot\boldsymbol{v} \tag{4.1.15}$$

where ***u*** and ***v*** also denote the vectors generating the corresponding matrices according to equation (4.1.11). The tensor (4.1.14) contains eight measurable quantities: λ, μ, and the two intrinsic vectors. Written at length, the entries of this tensor are of the form:

$$w_{ij} = \lambda u_i u_j + \mu v_i v_j - \frac{1}{2}(\lambda\boldsymbol{u}^2 + \mu\boldsymbol{v}^2)\delta_{ij}$$

or, symbolically: (4.1.16)

$$\boldsymbol{w} = \lambda\boldsymbol{u}\otimes\boldsymbol{u} + \mu\boldsymbol{v}\otimes\boldsymbol{v} - e\boldsymbol{1}$$

where ***u*** and ***v*** are vectors, and we used the following notations:

$$e \overset{def}{=} \frac{1}{2}(\lambda\boldsymbol{u}^2 + \mu\boldsymbol{v}^2) \quad \text{and} \quad \boldsymbol{g} \overset{def}{=} \sqrt{\lambda\mu}\cdot(\boldsymbol{u}\times\boldsymbol{v}) \tag{4.1.17}$$

It is easy to see that this tensor has three real eigenvalues, in general distinct. Indeed, its orthogonal invariants are:

$$I_1 = -e, \quad I_2 = -e^2 + \boldsymbol{g}^2, \quad I_3 = -e(e^2 - \boldsymbol{g}^2) \tag{4.1.18}$$

so that the eigenvalues of tensor $\boldsymbol{w}$ can then be calculated as the roots of the corresponding characteristic equation – the cubic equation having the invariants (4.1.18) as coefficients – and they are:

$$w_1 = e \quad \text{and} \quad w_{2,3} = \pm\sqrt{e^2 - \boldsymbol{g}^2} \tag{4.1.19}$$

It turns out that the pair from equation (4.1.17) gives one eigenvalue of $\boldsymbol{w}$ and the corresponding eigenvector. The other two eigenvectors of $\boldsymbol{w}$ are orthogonal, and located in the plane of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$: they are linear combinations of these two vectors.

At this point we have to give an explanation. Indeed, the general definition (4.1.13) of the tensor $\boldsymbol{w}$ involves quite a few quantities in order to be established by measurement: the constants α, β, γ, the lengths of the two vectors and their orientations, a total of nine quantities. However, this fact is only apparent, for we deal here with a symmetric matrix, having therefore only six independent components. As a matter of fact the representation in equation (4.1.16) has only eight quantities. As the three eigenvalues seem to be mandatory no matter how we proceed, for the two vectors only remains a need for only three quantities, leading us to the idea that three of the nine parameters are redundant. The problem popped up even from the pioneering works of Fresnel, in the form of representability of the elliptically polarized light. Its solution took different forms along the time leading eventually to the science of *ellipsometry*, whose first champions were apparently Stokes and Verdet [51]. Especially Émile Verdet insisted at length upon the statistical aspect of the problem which, according to any imaginable criterion, seems to be indeed its essential nature [50]. Here we give an inedited shade to this statistical aspect.

According to the above theory, the eigenvalues of our tensor $\boldsymbol{w}$, given in equation (4.1.19) are already statistical expressions based, like any such expressions, on some statistics in continua, called the *Novozhilov*'*s averages* ([36], Chapter 7, §7.5; see, for conformity, [38]):

$$w_n = \frac{1}{3}(w_1 + w_2 + w_3)$$
$$\text{and} \tag{4.1.20}$$
$$w_t^2 = \frac{1}{15}[(w_2 - w_3)^2 + (w_3 - w_1)^2 + (w_1 - w_2)^2]$$

The first of these statistics represent the projection of the vector having the components given by the eigenvalues $\langle w| \equiv (w_1, w_2, w_3)$ along the diagonal of an octant in a local reference frame, which is also the normal to the octahedral plane of the octant. As to the second statistics, it represents, up to the numerical factor, of course, the length of the component of the vector $|w\rangle$ in this octahedral plane. It is perhaps worth taking notice, while it is fresh here, of the fact that this statistical characterization is *eightfold*, for there are eight octants of the reference frame, with different signs of the components of the vector $|w\rangle$. This simply means that the linear coupling between the two resonators in order to offer the tensor $\boldsymbol{w}$ describing the light is eightfold. It might be refreshing for a classical physicist to learn that the classical physics of light established by Fresnel, naturally contains the modern eightfold way of the structure of matter: perhaps the quarks are not quite so strange after all, and they are, indeed, constructions of the mind allowing us to connect the observables, as one can often hear.

Using the eigenvalues (4.1.19), the two measurable statistical components of the tensor $\boldsymbol{w}$ from equations (4.1.20) are [36]:

$$\langle w|n\rangle = -\frac{2}{\sqrt{3}}e, \quad |w_t\rangle = \frac{2}{3}\begin{pmatrix} -2e \\ e+3\sqrt{e^2-\boldsymbol{g}^2} \\ e-3\sqrt{e^2-\boldsymbol{g}^2} \end{pmatrix} \tag{4.1.21}$$

As long as only the values (4.1.20) are measured, the orientation of the vector from (4.1.21) in the octahedral plane always remains undecided. This orientation is, again, out of our control *per se*, but it can be measured. It can be accounted for by an angle easy to measure in case we have a reference direction in the octahedral plane at our disposal. Assume, indeed, that we have such a reference, as given by a particular tensor of the form given in equation (4.1.15) with the characteristic vector $\boldsymbol{\xi}$ say. Then, for this tensor we have, with obvious notations:

$$\langle \xi|n\rangle = -\frac{1}{\sqrt{3}}\xi^2, \quad |\xi_t\rangle = \frac{2}{3}\xi^2\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \tag{4.1.22}$$

If the vector $|\xi\rangle$ is perpendicular on both $|u\rangle$ and $|v\rangle$, then the tensors $\boldsymbol{w}$ and $\boldsymbol{\xi}$ commute. Thus, they have a common reference frame and it can be arranged that their octahedral planes coincide. It is in this case that the direction of the vector from equation (4.1.22), which is fixed, can be correctly chosen as a reference direction in the octahedral plane. Then the angle ϕ of the vector (4.1.21) with respect to this fixed direction in the common octahedral plane can be calculated from a geometrical formula (*loc. cit. ante*), which here amounts to:

$$\cos\phi = -\frac{e}{\sqrt{4e^2 - 3\mathbf{g}^2}} \tag{4.1.23}$$

This shows that, under specified conditions, the angle ϕ is independent on the reference vector. With a proper choice of sign for the square root, the origin $\phi = 0(\mathrm{mod}2\pi)$ of this angle occurs only for the cases where $e = g$. This condition means, in turn, that the angle, θ say, between the vectors $|u\rangle$ and $|v\rangle$, calculated on the basis of the quantities from equation (4.1.17), is given by equation:

$$|\sin\theta| = \frac{1}{2}\left|\frac{\lambda u^2 + \mu v^2}{uv\sqrt{\lambda\mu}}\right| \tag{4.1.24}$$

As the quantity from the right-hand side here is always greater than or equal to *1*, the angle between vectors $|u\rangle$ and $|v\rangle$ cannot be but *90*°. Thus, the *initial condition* for the characteristic angle of tensor $\boldsymbol{w}$ in the octahedral plane takes place when the vector $|u\rangle$ is perpendicular to $|v\rangle$ and their plane is perpendicular to vector $|\xi\rangle$. If this last vector is given by a ray for instance, we have the classical image of the propagation according to Fresnel. One has to notice, however, that the price paid here for avoiding the classical kinematics in describing the vibratory motion, is accepting from the very beginning the planar description of the wave by *two* vectors whose physical meaning may be a challenge.

Regarding the problem of measurement, one can notice that it refers actually to just two quantities and an angle: anything else seems to be inference from these three quantities. The redundancy is due, as always in physics, to our geometrical models of reality: vectors and tensors. Mention should be made of the important fact that the perpendicularity of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is not a purely geometrical property, but the consequence of some preexistent statistics.

4.2. THE ELECTROMAGNETIC LIGHT

There is nothing more to say in order to see that the previous point of view was indeed "incarnated", as it were, in the ideas of James Clerk Maxwell: the tensor from equation (4.1.16) is plainly a *classical Maxwell stress tensor*, if for $\boldsymbol{u}$ and $\boldsymbol{v}$ we take the classical electromagnetic fields $\boldsymbol{e}$ and respectively $\boldsymbol{b}$. Then the parameters λ and μ can be taken to represent some measures which would indicate how much of this ether is space and how much is matter. At least this seems to be the conclusion of an exhaustive analysis [13], showing that at least one of these parameters *has to be taken as a density*. In a word, classically speaking, the equation (4.1.16) represents an *ether*: a *state of field cumulating matter and light properties*.

Now, if the light remains the same through vacuum, and one can imagine that the light is due to the motion, we have in the tensor (4.1.16) a representative of this light. The problem then arises as to the uniqueness of that tensor. We can formulate this issue as a problem: find the most general linear transformation of the vector fields ***e*** and ***b*** in their plane:

$$\boldsymbol{e}' = \alpha\boldsymbol{e} + \beta\boldsymbol{b}, \quad \boldsymbol{b}' = \gamma\boldsymbol{e} + \delta\boldsymbol{b} \tag{4.2.1}$$

which preserves the Maxwell stress tensor. Here the notations ***e*** and ***b*** aim to suggest that we have to do with a kind of electromagnetic fields, as in the classical case. Rewriting the tensor of Maxwell stresses in this connotation, we have:

$$t_{ij} = \lambda e_{ij} + \mu b_{ij}, \quad e_{ij} \overset{def}{=} e_i e_j - \frac{1}{2}e^2\delta_{ij}, \quad b_{ij} \overset{def}{=} b_i b_j - \frac{1}{2}b^2\delta_{ij} \tag{4.2.2}$$

Any kind of invariance of this tensor would necessarily lead to a connection between the parameters λ, μ and the entries of the matrix from equation (4.2.1), which allow us a concrete description – and a solution, hopefully – of the modern problem of *vacuum tunneling* [29]: the fields are changed by the presence of matter in ether, in order to adapt themselves to the different *local* properties represented by the parameters λ and μ. What remains to be decided is how do we define the invariance of the tensor (4.2.2), and a proposal presents itself just naturally: the entries of the matrix ***t*** *have to remain unchanged*. Then, *a fortiori* all of the invariants of this tensor remain the same and, therefore, what is measured out of it has the same value for the whole coordinate space of definition for this tensor.

This proposal comes out from a twofold suggestion: first, is the importance of the tensor ***t*** in general relativity and, secondly, we entertain the belief that what we are locally recording, is what has been happening far away in our space and, therefore, our conclusions regarding the structure of this universe, based on this recording, are the right ones. There is, however, a more subtle reason for this kind of "conservation law": if the background radiation has a Planck spectrum [14], then we can say that the part of the universe we inhabit is a Wien-Lummer cavity. In view of the scale invariance of the Planck's spectrum, each and every one of these cavities should behave the same way, and the conservation of the tensor ***t*** appears as the only possibility of defining the equilibrium temperature for radiation, according to the Planck's idea. Implicitly then, the conclusion is valid for the Procopiu's quantization procedure.

Thus, if by the transformation (4.2.1) the fields ($\boldsymbol{e}'$,$\boldsymbol{b}'$) are to be found in an environment described by (λ', μ'), then the conservation: $t_{ij} = t'_{ij}$ can be transcribed as:

$$(\alpha\delta - \beta\gamma)\sqrt{\lambda\mu} = \sqrt{\lambda'\mu'}, \qquad \alpha\beta\lambda + \gamma\delta\mu = 0$$
$$\alpha^2\lambda + \gamma^2\mu = \lambda', \qquad \beta^2\lambda + \delta^2\mu = \mu' \tag{4.2.3}$$

In order to learn how to use these equations, let us just find some particular solutions of the system (4.2.3).

For the special case of *homogeneity of the vacuum* we have: $\lambda = \lambda'$, $\mu = \mu'$. The first of the equations (4.2.3) then shows that the transformation (4.2.1) is unimodular. From an algebraic point of view, the last three equations then form a separate homogeneous system, and thus the system (4.2.3) is equivalent to:

$$\alpha = \delta, \quad \beta = -\frac{\mu}{\lambda}\gamma, \quad \alpha^2 + \frac{\mu}{\lambda}\gamma^2 = 1 \tag{4.2.4}$$

The last of these equations shows that we can express the two entries α and γ trigonometrically, *via* an arbitrary phase parameter, ϕ say, in the form:

$$\alpha = \cos\phi, \quad \gamma = \sqrt{\frac{\lambda}{\mu}}\sin\phi \tag{4.2.5}$$

In these cases, the transformation (4.2.1) that does not change the Maxwell stress tensor is realized by the matrix of unit determinant:

$$\begin{pmatrix} \cos\phi & -n\sin\phi \\ \sin\phi/n & \cos\phi \end{pmatrix}, \quad n \equiv \sqrt{\frac{\mu}{\lambda}} \tag{4.2.6}$$

In general, we may accept a more relaxed condition for the vacuum, equivalent, in a way, with the fact that the "refraction index" is constant. Such a condition amounts to:

$$\frac{\mu}{\lambda} = \frac{\mu'}{\lambda'} \equiv n^2 \tag{4.2.7}$$

It means matter non-homogeneity in regards to physical properties, although when the physics is referred to a "refraction index" n, the matter is actually homogeneous. In this case, the system (4.2.3) gives the matrix of transformation in equation (4.2.1) as:

$$\sqrt{m}\cdot\begin{pmatrix} \cos\phi & -n\sin\phi \\ \sin\phi/n & \cos\phi \end{pmatrix}, \quad m\equiv\frac{\lambda'}{\lambda} \tag{4.2.8}$$

Formally, this matrix is not different from that gotten in equation (4.2.6): it is only that it does not have unit determinant. Let us work on this last matrix, in order to build a significant geometry here. Operating the transformation of parameters:

$$u = n\cot\phi, \quad v = \frac{n}{\sin\phi} \tag{4.2.9}$$

we can cast the matrix into the parametric form:

$$\boldsymbol{m}\equiv\frac{\sqrt{m}}{v}\begin{pmatrix} u & u^2-v^2 \\ 1 & u \end{pmatrix} \tag{4.2.10}$$

In order to reckon what to make out of this matrix, we need to know what to make out of the transformation (4.2.1) itself. Thus, if we differentiate that equation, we get:

$$\begin{pmatrix} d\boldsymbol{e}' \\ d\boldsymbol{b}' \end{pmatrix}=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\cdot\begin{pmatrix} d\boldsymbol{e} \\ d\boldsymbol{b} \end{pmatrix}+\begin{pmatrix} d\alpha & d\beta \\ d\gamma & d\delta \end{pmatrix}\cdot\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{b} \end{pmatrix} \tag{4.2.11}$$

On this occasion it is worth our while simplifying the notation, by adopting one which is kind of self-explanatory when we avail ourselves of a Dirac's notation. So, we are transcribing the equation (4.2.11) as:

$$|d\boldsymbol{e}'\rangle=\boldsymbol{m}\cdot|d\boldsymbol{e}\rangle+d\boldsymbol{m}\cdot|\boldsymbol{e}\rangle \quad \therefore \quad \boldsymbol{m}^{-1}\cdot|d\boldsymbol{e}'\rangle=|d\boldsymbol{e}\rangle+\boldsymbol{m}^{-1}\cdot d\boldsymbol{m}\cdot|\boldsymbol{e}\rangle \tag{4.2.12}$$

Assume a state of the fields where $|d\boldsymbol{e}\rangle=|\boldsymbol{0}\rangle$: we take it as *a static state*, if the time is encompassing all the possible variations of a field magnitude. This static state is "propagated" as in transformation (4.2.1), and the medium transforms it into a dynamic state, just by adding a contribution to field, generated by the matrix of propagation $\boldsymbol{m}$. This contribution amounts to $\boldsymbol{m}^{-1}\cdot d\boldsymbol{m}\cdot|\boldsymbol{e}\rangle$, where:

$$\boldsymbol{m}^{-1}\cdot d\boldsymbol{m} = d\xi\cdot\boldsymbol{1}+\boldsymbol{\omega},\quad \boldsymbol{\omega}=\begin{pmatrix} -\omega^2/2 & -\omega^3 \\ \omega^1 & \omega^2/2 \end{pmatrix} \tag{4.2.13}$$

Here $\omega^{1,2,3}$ are the three differential 1-forms, components of the **sl**(2, R) coframe which describes an *instanton* [see [35]; §4.4, equation (4.4.3)], and we used the notation: $2\xi = \ln(\det \boldsymbol{m})$. So, the fields $\boldsymbol{m}^{-1}\cdot|d\boldsymbol{e}'\rangle$ must be considered as "instantaneous fields" obtained from the static ones just by propagation. A classical counterpart of them is known in a particular illuminating occurrence.

Assuming here an interpretation by static ensembles, made possible as ensembles of equilibrium with static Newtonian force fields, the suggestion presents itself that the motion of matter through ether brings a rotation acting upon these force fields [25]. It is this experience which further shows that the electric and magnetic static forces act in a "tandem", so to speak, as a force whose expression is linear in the electric and magnetic fields, involving also linearly the two kinds of charges, electric and magnetic:

$$\boldsymbol{F}_{st} = q_e\boldsymbol{e} + q_m\boldsymbol{b} \tag{4.2.14}$$

These forces characterize equilibrium ensembles, whereby the particles possessing charges are in a *stationary state*. Assume, then, that a *state of motion* is described by a "Lorentz-transformed force", involving the *static force* from (4.2.14) and *a rotated counterpart*, with *the rotation defined by the static charges*:

$$\boldsymbol{F} = q_e\boldsymbol{e} + q_m\boldsymbol{b} + \frac{1}{c}\boldsymbol{v}\times(q_e\boldsymbol{b} - q_m\boldsymbol{e}) \tag{4.2.15}$$

while the equations describing the fields are "symmetric", *i.e.*, according to the Maxwell's idea, we have:

$$\begin{aligned} &\nabla\cdot\boldsymbol{e} = 4\pi q_e\rho,\quad \nabla\times\boldsymbol{e} = -\frac{1}{c}\frac{\partial\boldsymbol{b}}{\partial t} - \frac{4\pi}{c}q_m\boldsymbol{j} \\ &\nabla\cdot\boldsymbol{b} = 4\pi q_m\rho,\quad \nabla\times\boldsymbol{b} = \frac{1}{c}\frac{\partial\boldsymbol{e}}{\partial t} + \frac{4\pi}{c}q_e\boldsymbol{j} \end{aligned} \tag{4.2.16}$$

Here, ρ is the numerical density of particles, while $\boldsymbol{j}$ is their current. These equations have the virtue of reducing themselves to the usual Maxwell equations for either $q_m = 0$ or $q_e = 0$. Notice, however, that with no such quantitative consideration on charges – which is quite particular, and, therefore, from Katz's natural-philosophical point of view regarding the charges themselves, should be, in a way, irrelevant – we can define two new field variables *via* the *genuine rotation* generated by the two charges:

$$e\boldsymbol{E} = q_e\boldsymbol{e} + q_m\boldsymbol{b}, \quad e\boldsymbol{B} = -q_m\boldsymbol{e} + q_e\boldsymbol{b}, \quad e^2 = q_e^2 + q_m^2 \tag{4.2.17}$$

and with these fields the force (4.2.15) becomes *the Lorentz force* as we usually know it from classical electrodynamics:

$$\boldsymbol{F} = e\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \tag{4.2.18}$$

while the Maxwell equations become those we know from the textbooks:

$$\begin{gathered} \nabla \cdot \boldsymbol{E} = 4\pi e\rho, \quad \nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c} e \cdot \boldsymbol{j} \end{gathered} \tag{4.2.19}$$

However, while in the first symmetric version, the rotation is determined by the ratio of charges, which in turn needs a special natural philosophy involving these charges (see [30]; see also [34], §3.1), in the Lorentz version, the theory is pending on a genuine *space rotation* that needs *central forces acting sideways*. This notion may seem contradictory, but we use it nevertheless in order to pinpoint a fact of which we need to account theoretically.

Namely, insofar as a force is created by a physical characteristic of a particle – specifically, mass and charges – it is, no doubt, central: the particle creating it is the obvious center of force. On the other hand, when it comes to the action of such a force, it can be twofold: the force can act along the direction to the particle that created it, or across this direction, *i.e. sideways*, with an expression of J. J. Thomson. Besides the fact that, at first sight, this concept appears as strange by itself, from the point of view of motion it requires a special arena where *the forces have to be logarithmic* (see [34]; Chapter 6, §6.2). This arena cannot be but the Louis de Broglie's region that we have found "strange" ([34], Chapter 2), which is an expression of the holographic property of light. This requirement leads to the necessity of a wave image, as de Broglie's theory stipulates, but it turns out to be valid along with the Maxwellian electrodynamics, just as Lorentz intended to show in the first place. The bottom line, then, is that the relativity, as an expression of the necessity of interpretation, needed the concept of wave: otherwise, the interpretation itself, as a necessary step in the construction of a theory of physical structures, could not be a full concept.

The fields defined by equation (4.2.12), starting from a static state which reflects the equilibrium of the ensembles of particles, are generalization of those obtained by the above charge-induced pure rotation in the classical electrodynamics. Thus, the matrix (4.2.13) accomplishes a *genuine duality transformation* of the static fields, not just a duality rotation, but a more involved transformation that can be written in the form:

$$\left|d\boldsymbol{E}\right\rangle = (\boldsymbol{m}^{-1}\cdot d\boldsymbol{m})\cdot\left|\boldsymbol{e}\right\rangle \quad \therefore \quad \left|d\boldsymbol{E}\right\rangle = (d\xi)\cdot\left|\boldsymbol{e}\right\rangle + \boldsymbol{\omega}\cdot\left|\boldsymbol{e}\right\rangle \tag{4.2.20}$$

In calculating these fields for the matrix ***m*** given in equation (4.2.10), we just have to calculate the matrix from (4.2.20).

The bottom line here is that the category of light, unlike the category of matter, is physically called to decide on the phases in matter: the matrix ***m*** from equation (4.2.10) is the one serving in indexing the geodesics in the nuclear matter (§3.4)

5. CONCLUSIONS

The resonator is a universal fundamental structure entering the physical structures of the two categories involved in the quantization process: the light and the matter. It is a dipole of charges, defined first by Max Planck in order to carry out the quantization of light. The Planck's resonator is electric dipole. In order to carry on the quantization in matter, the only existing procedure coping with that of Planck is the Procopiu quantization, and asks for a resonator defined as a magnetic dipole. Our results can be summarized as follows:

1) the optical medium accepting dipoles as fundamental constitutive structures is the so-called *Maxwell fish-eye*. This structure can be gotten as a Cayley-Klein, or absolute geometry, describing a charge continuum according to Katz's natural philosophy of charge.

2) the Maxwell fish-eye is a *holographic universe*, assuming the holography defined by coherence properties in the spirit of initial ideas of Dennis Gabor.

3) the nuclear matter can be described as a holographic universe that can be realized as such only with the help of the properties of light. The fields characterizing light are generalizing the classical Maxwellian fields, and are close to the modern *Yang-Mills fields*.

4) our analysis indicates that the *eightfold way* is a universal idea in theoretical physics. In order to understand the deep meaning of this statement, an observation may be in order: *the very Fresnel's physical theory of light is an expression of the eightfold way*.

Acknowledgments. Thanks are kindly due to Mrs. Cristina Irimia for gracefully handling the editing issues, which sometimes come so hard on an author.

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