

**NON-LINEAR FRICTION-TYPE BEHAVIORS IN LIQUID CRYSTALS
IN A MULTIFRACTAL PARADIGM OF MOTION**

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Fractal analysis of liquid crystals is usually and mainly focused on their intrinsic (structural) properties, and less on their dynamic (functional) properties. In the present paper, a new model, based on a multifractal structural-functional synergy for describing liquid crystals dynamics, is proposed. Then, liquid crystal entities dynamics will be explained in relation with the Scale Relativity Theory through continuous and non-differentiable (multifractal) curves, described with Schrödinger's theory and the hydrodynamic theory. Examining the hidden symmetry of the friction tensor at different scale resolutions, in the Madelung framework, synchronization modes among the liquid crystals entities are found through $SL(2, \mathbb{R})$ transformations groups. As such, at various scale resolutions, Riemannian manifolds with hyperbolic metrics, generated by means of direction parallel transport (in the Levi-Civita sense) can be defined. Different non-stationary friction characteristics can thus be highlighted at various scale resolutions, using harmonic mappings from the ordinary space to the hyperbolic one. Of these, the cellular and channel-type ones become fundamental.

Keywords: liquid crystals, fractal analysis, multifractal friction tensor, Scale Relativity Theory

INTRODUCTION

Liquid crystals (LCs) are common examples of partially ordered materials that can freely translate as liquid. They are ordered above a critical concentration or below a critical temperature. LCs have numerous general applications in various domains: materials design, nanoscience, biophysics, etc. Due to their anisotropy, LCs display optical, anisotropic mechanical and rheological properties [1,2]. Therefore, LCs are highly suitable for commercial applications, the most common being in the LC display industry [3]. One should also have in mind that a liquid crystal is a typical complex fluids system, which permits the development of new theoretical approaches beyond the specific fields of LCs, such as elastomers and colloid science, surface/ interfacial phenomena, polymers, and active matter [4].

Liquid crystals are mesophases between anisotropic crystalline (Fig. 1a) and isotropic liquids (Fig. 1e). The three major classes of LCs are: smectic, cholesteric, and nematic [5]. Nematics is the simplest one (Fig. 1d), exhibiting a long-range orientational order, *i.e.*, the molecules align almost parallel to each other, but there is no long-range correlation for the positions of the molecular center of mass. The cholesteric (Fig. 1.1(c)) and nematic orders are similar at local scale while, at larger scales, the director helix of cholesteric molecules has a spatial period. A nematic liquid crystal is a cholesteric liquid crystal without a helix. Figure 1.1(b) shows that smectics have a degree of translational ordering, resulting in a layer-like structure. Due to this partial translational ordering, the smectic phases are more viscous and closer to the crystalline than either the nematic or cholesteric phases [6].

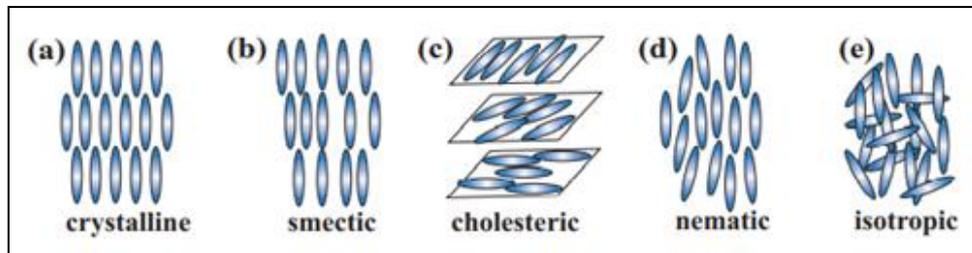


Figure 1. Crystalline (a), smectic (b), cholesteric (c), nematic (d), and isotropic (e) phases with rod-like molecules.

Standard models used for describing the dynamics of liquid crystals are based either on a combination of basic theories, derived especially from physics, or on computer simulations. Such descriptions usually involve both computations based on algorithms and simulations, as well as new developments of standard theories. Regarding the models developed based on standard theories, the next classes can be established:

i. Differentiable models, constructed on spaces with integer dimensions (*e.g.*, vector model, molecular model and Q-tensor model). The first type is the vector model, considering the Oseen-Frank theory [7,8] and Ericksen's theorem [9,10]. According to the vector model, the alignment of LC molecules at each material point x is assumed to follow a locally preferred direction $n(x) \in S^2$ (the unit sphere in three-dimensional space). The LC community has widely applied the vector theory, due to its simplicity; although this setting is approximate, it works quite well in many situations. The disadvantage of vector theories, however, is that they do not account for the head-to-tail symmetry of the rod-like molecular pattern where $-n$ should be equivalent to n [11]. The result may be an incorrect description of some systems, especially when defects exist. To study LC flow, Doi [12] further developed the molecular model proposed by Onsager [13] to characterize the nematic-isotropic phase transition. According to this theory, an orientational distribution function $f(x, m)$ describes the alignment behavior.

At a material point x , the distribution function represents the numerical density of molecules with orientation $m \in S^2$. Molecular models provide a more accurate description of molecular alignment, since f contains a higher amount of information about the alignment.

As the problem is usually multi-sided, the computation costs are often high. According to Landau-de Gennes' (LdG's) Q-tensor model [1], alignment of LC molecules at x -position is presented as a 3×3 symmetric matrix without trace $Q(x)$. Identifying the Q-tensor order parameter from the orientational distribution function $f(x, m)$ relates the order parameter to the second moment. It is not assumed that the molecular alignment has a preferred direction, so it can describe biaxiality. It can be asserted that the vector and tensor theories are macroscopic theories, since they are derived from continuum mechanics, whereas the molecular theory is microscopic and originates from statistical mechanics. Despite the different physical perspectives from which they have been approached, they have all played a significant role in the study of LCs. To understand LC analysis, it is important to know these models and their relationships. Additionally, the coefficients in macroscopic theories are determined phenomenologically, making unclear their interpretation in terms of basic physical measurements. When examining their relationship, these coefficients can be interpreted in terms of molecular parameters, rather than phenomenologically.

ii. Two approaches for calculating stable fault models can be applied from a numerical perspective (computational models), one of which can be based on the minimization of energy [14–18]. Other approaches focus on the dynamics of free energy-driven gradient flows in response to individual LC models [19–23]. The solution of gradient flow equations can be achieved by several efficient numerical procedures, including energy-stable methods, such as the convex division [24], and the scalar auxiliary variable ones [25]. Also, new approaches to soft matter, including LCs, are being explored with machine learning [26]. Furthermore, the numerical model of LC hydrodynamics, simulating LC flows, LC droplets, LC colloidal composites, etc. has been extensively developed [27,28].

iii. Fractal models. Fractal geometry has been used in various fields, such as physics, chemistry, material science, biology, geology, etc. It is more commonly employed in complex and disordered systems to explain various modeling, growth, and aggregation processes [29]. The phase formation processes in soft condensed matter (colloids and polymers) generate very complex structures that can be described by fractal geometry, while the fractal dimension obtained can be related to simple generic growth models, like diffusion limited aggregation, cluster-cluster aggregation, or percolation [29]. Considering this, worth noting is that only a few studies on the relationship between fractal geometry and liquid crystals have been published. In one of them, the fractal dimension of a single dendritic-like texture of a discotic columnar hexagonal phase was calculated [30]. Other studies have described fractally homogeneous nuclei distributions in nematic liquid crystals

during phase ordering and coarsening [31,32]. Finally, the author summarized thorough investigations into the fractal characteristics of liquid crystal phase ordering in bent-core molecules in [29].

iv. The Scale Relativity Theory has resulted in new advancements, either in monofractal dynamics, as in the case of Nottale [33], or in multifractal dynamics, as in the case of the Multifractal Theory of Motion, expanding the earlier class of models [34,35].

In the Scale Relativity Theory in the sense of Nottale [33], as well as in the Multifractal Theory of Motion [34,35], the fundamental hypothesis is the following: any type of liquid crystal is assimilated both structurally and functionally to a multifractal object. Motions on continuous and non-differentiable curves can be used to represent the dynamics of liquid crystal entities (multifractal curves).

Let us consider the following scenario: between two successive interactions of entities belonging to any liquid crystal, the trajectory of entities is a straight line that becomes non-differentiable at the impact point. Taking into account that all contact points constitute an uncountable collection of points, the trajectories of the liquid crystal entities become continuous and non-differentiable (*i.e.*, fractal curves). Evidently, reality is much more complicated, considering both the diversity of the entities which compose any liquid crystal and the various interactions between them in the form of double interactions, etc. Any sort of liquid crystal can be assimilated to a multifractal by extrapolating the prior rationale.

All these considerations imply that, in the description of the liquid crystal dynamics in its various presentation forms, instead of “working” with a single variable (regardless of its nature, *i.e.*, velocity, density, etc.) described by a strict non-differentiable function, it is possible to “work” only with approximations of this mathematical function, obtained by averaging them on different scale resolutions. As a consequence, any variable aimed at describing liquid crystal dynamics in its various forms will perform as the limit of a family of mathematical functions, this being non-differentiable for null scale resolutions and differentiable otherwise [33,35]. In other words, from a mathematical point of view, these variables can be explained through multifractal functions (*i.e.*, functions dependent not only on spatial and temporal coordinates, but also on scale resolution).

The deterministic trajectories of any structural unit belonging to the liquid crystal in its various presentation forms can be replaced by a collection of potential (“virtual”) trajectories for a large temporal scale resolution with respect to the inverse of the highest Lyapunov exponent [36–38]. Then, the concept of definite trajectory in the description of liquid crystal dynamics in its various presentation forms can be substituted by that of probability density.

Considering all the above observations, multifractality expressed through stochasticity in the description of liquid crystal dynamics in its various presentation forms becomes operational in the multifractal paradigm through the Multifractal Theory of Motion.

The present analysis aimed at modeling the behavior of liquid crystals, specifically friction phenomena. A mathematical model was created considering a liquid crystal as a multifractal object, its dynamics being analyzed in relation with the Scale Relativity Theory, using various operational procedures (group invariances, variational principles, harmonic mappings, etc.).

MATHEMATICAL MODEL

CONSEQUENCES OF NON-DIFFERENTIABILITY IN A MULTIFRACTAL PARADIGM

In what follows, let us assume that any liquid crystal can be assimilated, both structurally and functionally, to a multifractal object. The motions of the entities of any liquid crystals can then be described by continuous and non-differentiable curves in the context of the Scale Relativity Theory (multifractal curves). Such a “non-differentiable” procedure to approach these motions has important consequences, such as:

Any multifractal curve which describes the dynamics of any liquid crystals entities is explicitly scale δt dependent. In such a context, its length tends to infinity when δt tends to zero (Lebesgue theorem). Moreover, the space of any liquid crystal dynamics becomes a multifractal in Mandelbrot sense;

The dynamics of any liquid crystals is related to the behavior of a set of functions during the zoom operation of δt , as such, $\delta t \equiv dt$ through the functionality of substitution principle;

The dynamics of any liquid crystals is described through multifractal variables (for example, velocity, density, etc.). Then, two derivatives of any variable field $Q(t, dt)$ which describe the liquid crystals dynamics can be defined:

$$\begin{aligned} \frac{dQ_+}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{Q(t, t + \Delta t) - Q(t, \Delta t)}{\Delta t}, \\ \frac{dQ_-}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{Q(t, \Delta t) - Q(t - \Delta t, \Delta t)}{\Delta t}. \end{aligned} \quad (1)$$

The sign “+” corresponds to the forward dynamics of the liquid crystals, while the sign “-” corresponds to the backward ones;

The differential of the spatial coordinate field which describes the dynamics of any liquid crystals entities has the form:

$$d_{\pm} X^i(t, dt) = d_{\pm} x^i(t) + d_{\pm} \xi(t, dt) \quad (2)$$

The differentiable part, $d_{\pm} X^i(t)$, is scale resolution independent, while the non-differentiable part, $d_{\pm} \xi(t, dt)$, is scale resolution dependent.

The non-differentiable part of the spatial coordinate field, which describes the dynamics of any liquid crystals entities, satisfies the non-differentiable multifractal equation [35]:

$$d_{\pm}\xi^i(t, dt) = \lambda_{\pm}^i(dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \quad (3)$$

where λ_{\pm}^i are constant coefficients associated to differentiable – non-differentiable scale transition, $f(\alpha)$ is the singularity spectrum of order α of fractal dimension D_F , and α is the singularity index.

There are various modes, and hence a wide range of fractal dimensional definitions, to characterize the dynamics of any liquid crystal entity: fractal dimension in the sense of Kolmogorov, fractal dimension in the sense of Hausdorff – Besikovich, and so on. When using one of these definitions in the dynamics of any liquid crystalline entity, the fractal dimension value must remain constant and arbitrary throughout the dynamical analysis: regularly found are $D_F < 2$ for correlative processes, and $D_F > 2$ for non-correlative processes, etc. In such a conjecture, through (3) it is possible to identify not only the “areas” of the dynamics of any liquid crystals entities characterized by a certain fractal dimension (*i.e.*, in the case of monofractal dimensions), but also the number of “areas” whose fractal dimensions are situated in an interval of values (*i.e.*, in the case of multifractal dimensions). More than that, through the singularity spectrum $f(\alpha)$, it is possible to identify classes of universality in the dynamics of any liquid crystals entities laws, even when regular or strange attractors have different aspects;

The differential time reflection invariance of any variable which describes the dynamics of any liquid crystals entities is recovered by means of operator:

$$\frac{\tilde{d}}{dt} = \frac{1}{2} \left(\frac{d_+ + d_-}{dt} \right) - \frac{i}{2} \left(\frac{d_+ - d_-}{dt} \right). \quad (4)$$

This is a natural result of Cresson’s theorem. For example, applying operator (4) to X^i yields the complex velocity fields:

$$\hat{v}^i = \frac{\tilde{d}X^i}{dt} = V_B^i - V_F^i \quad (5)$$

with:

$$V_B^i = \frac{1}{2} \frac{d_+ X^i + d_- X^i}{dt}, \quad V_F^i = \frac{1}{2} \frac{d_+ X^i - d_- X^i}{dt}, \quad i = 1, 2, 3 \quad (6)$$

The real part V_B^i (differential velocity) is scale resolution independent, while the imaginary one V_F^i (non-differentiable velocity) is scale resolution dependent;

Since multifractalization, which describes the dynamics of any liquid crystals entities, implies stochasticization (see Section 1), the whole statistic “arsenal” in the form of averages, variances, covariances, etc. becomes operational. Thus, the following functionality will be chosen for the average of $d_{\pm} X^i$:

$$\langle d_{\pm} X^i \rangle \equiv d_{\pm} x^i, \quad (7)$$

with:

$$\langle d_{\pm} \xi^i \rangle = 0. \quad (8)$$

The previous relation (8) implies that the average of the non-differential part of the spatial coordinate field is null;

The dynamics of any liquid crystals' entities can be described through the scale covariant derivative given by operator:

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^i \partial_i + \frac{1}{4} (dt) \left[\frac{2}{f(\alpha)} \right]^{-1} D^{ik} \partial_i \partial_k, \quad (9)$$

where:

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial X^i}, \quad \partial_i \partial_k = \frac{\partial^2}{\partial X^i \partial X^k}. \quad (10)$$

Numerous types of multifractalization *via* stochasticization processes are possible, the Markovian and non-Markovian stochastic processes being among the most frequently used ones. Only multifractalization through Markovian stochastic processes, *i.e.*, those described by constraints, will be treated in the following, specifically in the description of liquid crystal entity behavior:

$$\lambda_+^i \lambda_+^l = \lambda_-^i \lambda_-^l = 2\lambda \delta^{il} \quad (11)$$

and:

$$f(\alpha) \equiv D_F \quad (12)$$

where λ is a specific coefficient associated to the fractal-non-fractal scale transition and δ^{il} is Kronecker's pseudo-tensor. With these constraints, the scale covariant derivative (9) becomes:

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^l \partial_l - i\lambda (dt) \left[\frac{2}{D_F} \right]^{-1} \partial_l \partial^l \quad (13)$$

In the particular case of Peano-type curves, which implies $D_F = 2$, the scale covariant derivative (13) takes the standard form from the Scale Relativity Theory:

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^l \partial_l - iD \partial_l \partial^l \quad (14)$$

where $\lambda \equiv D$ is the diffusion coefficient associated to fractal-non-fractal scale transition.

Therefore, this model generalizes all results of Nottale's theory (*i.e.*, the Scale Relativity Theory). Moreover, for Compton scale resolution, (14) becomes a "quantum operator".

In what follows, since the dynamics of any liquid crystal entity is analyzed on a multifractal manifold, all variables and parameters employed must be understood in the sense of the multifractal concept.

DYNAMICS IN LIQUID CRYSTALS THROUGH SCHRÖDINGER AND HYDRODYNAMIC
“REGIMES” AT VARIOUS SCALE RESOLUTIONS.
THE MULTIFRACTAL FRICTION TENSOR

Accepting the functionality of the scale covariance principle, *i.e.*, applying operator (9) to the complex velocity fields (5) in the absence of any external constraint, the motion equations (the geodesics equation) that describe the dynamics of any liquid crystals entities take the form:

$$\frac{d\hat{V}^i}{dt} = \partial_t \hat{V}^i + \hat{V}^l \partial_l \hat{V}^i + \frac{1}{4} (dt) \left[\frac{2}{f(\alpha)} \right]^{-1} D^{lk} \partial_l \partial_k \hat{V}^i = 0. \quad (15)$$

This means that the multifractal acceleration, for any dynamics of any liquid crystal entities, $\partial_t \hat{V}^i$, the multifractal convection, $\hat{V}^l \partial_l \hat{V}^i$ and the multifractal dissipation (multifractal internal friction) $D^{lk} \partial_l \partial_k \hat{V}^i$ make their balance in any point of the curve. Particularly for (11), the motion equations (the geodesics equation) (15) become:

$$\frac{d\hat{V}^i}{dt} = \partial_t \hat{V}^i + \hat{V}^l \partial_l \hat{V}^i - i\lambda (dt) \left[\frac{2}{f(\alpha)} \right]^{-1} \partial_l \partial^l \hat{V}^i = 0 \quad (16)$$

Now, separating the dynamics of any liquid crystals' entities on scale resolutions (the differentiable and non-differentiable scale resolutions), (15) becomes:

$$\begin{aligned} \partial_t V_D^i + V_D^l \partial_l V_D^i - V_F^l \partial_l V_F^i + \frac{1}{4} (dt) \left[\frac{2}{f(\alpha)} \right]^{-1} D^{lk} \partial_l \partial_k V_D^i &= 0 \\ \partial_t V_F^i + V_F^l \partial_l V_D^i + V_D^l \partial_l V_F^i + \frac{1}{4} (dt) \left[\frac{2}{f(\alpha)} \right]^{-1} D^{lk} \partial_l \partial_k V_F^i &= 0, \end{aligned} \quad (17)$$

while (16) takes the form:

$$\begin{aligned} \partial_t V_D^i + V_D^l \partial_l V_D^i - \left[V_F^l + \lambda (dt) \left[\frac{2}{f(\alpha)} \right]^{-1} \partial^l \right] \partial_l V_F^i &= 0 \\ \partial_t V_F^i + V_D^l \partial_l V_F^i + \left[V_F^l + \lambda (dt) \left[\frac{2}{f(\alpha)} \right]^{-1} \partial^l \right] \partial_l V_D^i &= 0. \end{aligned} \quad (18)$$

For irrotational motions of the dynamics of any liquid crystals' entities, the multifractal complex velocity field (5) becomes:

$$\hat{v}^i = -2i\lambda(dt)\left[\frac{2}{f(\alpha)}\right]^{-1}\partial^i \ln \Psi \quad (19)$$

where:

$$\chi = -2i\lambda(dt)\left[\frac{2}{f(\alpha)}\right]^{-1} \ln \Psi \quad (20)$$

is the multifractal complex scalar potential of the multifractal complex velocity field (5) and Ψ is the multifractal states function (with the significances of Ψ). The geodesics equation (16) assumes the form of a multifractal Schrödinger equation in these conditions, where (19) is substituted for (16):

$$\lambda^2(dt)\left[\frac{4}{f(\alpha)}\right]^{-2}\partial^i\partial_i\Psi + i\lambda(dt)\left[\frac{2}{f(\alpha)}\right]^{-1}\partial_t\Psi = 0 \quad (21)$$

As a result, the dynamics of every liquid crystal entity can be characterized using multifractal complex velocity fields (19) through multifractal Schrödinger "regimes" (*i.e.*, Schrödinger-type equations at various scale resolutions).

Furthermore, if Ψ is selected in the form:

$$\Psi = \sqrt{\rho}e^{is}, \quad (22)$$

where $\sqrt{\rho}$ is the amplitude and s is the phase, the multifractal complex velocity fields (19) are then expressed explicitly as:

$$\hat{v}^i = 2\lambda(dt)\left[\frac{2}{f(\alpha)}\right]^{-1}\partial^i s - i\lambda(dt)\left[\frac{2}{f(\alpha)}\right]^{-1}\partial^i \ln \rho \quad (23)$$

which implies the multifractal real velocity fields:

$$V_D^i = 2\lambda(dt)\left[\frac{2}{f(\alpha)}\right]^{-1}\partial^i s \quad (24)$$

$$V_F^i = \lambda(dt)\left[\frac{2}{f(\alpha)}\right]^{-1}\partial^i \ln \rho. \quad (25)$$

By (22), (24) and (25) and using the mathematical procedures from [34], the geodesics multifractal hydrodynamic equations (21) appear as:

$$\partial_t V_D^i + V_D^l \partial_l V_D^i = -\partial^i Q \quad (26)$$

$$\partial_t \rho + \partial_i (\rho V_D^i) = 0 \quad (27)$$

with Q the multifractal specific potential:

$$Q = -2\lambda^2(dt)\left[\frac{4}{f(\alpha)}\right]^{-2}\frac{\partial^i\partial_i\sqrt{\rho}}{\sqrt{\rho}} = -V_F^i V_F^i - \frac{1}{2}\lambda(dt)\left[\frac{2}{f(\alpha)}\right]^{-1}\partial_i V_F^i. \quad (28)$$

The multifractal specific momentum conservation law is represented by equation (26), and the multifractal states density conservation law by equation (27), respectively. The equation for the multifractal particular force is derived using the multifractal specific potential (28):

$$F^i = -\partial^i Q = -2\lambda^2 (dt)^{\left[\frac{4}{\bar{r}(\bar{\alpha})}\right]-2} \partial^i \frac{\partial^i \partial_i \sqrt{\rho}}{\sqrt{\rho}} \quad (29)$$

being a measure of the multifractality of the motion curves.

As a result, the dynamics of every liquid crystal is defined by multifractal hydrodynamic “regimes” for multifractal complex velocity fields (23). (*i.e.*, hydrodynamic-type equations at various scale resolutions). In this last context, the following consequences will result:

Through the multifractal specific force, any liquid crystal entities are constantly in contact with a multifractal medium (29).

A multifractal fluid whose dynamics is characterized by the multifractal hydrodynamic model [see Eqs. (26) – (28)] can be used to identify any liquid crystal.

Although the velocity field V_F^i induces non-manifest dynamics, it also facilitates in the transmission of multifractal specific momentum and multifractal energy of focus. This is due to the absence of V_F^i from the density conservation law derived from multifractal states.

The “self” aspect of the multifractal specific momentum transfer should be taken into account in every interpretation of the multifractal specific potential. While the multifractal energy is stored in the form of mass motion and multifractal potential energy, some part of it is available elsewhere and only the total is conserved. The reversibility and existence of eigenstates are guaranteed by the conservation of multifractal energy and multifractal momentum, while denying a multifractal Lévy motion force of interaction with a multifractal external medium.

If using the tensor:

$$\hat{t}^{ii} = 2\lambda^2 (dt)^{\left[\frac{4}{\bar{r}(\bar{\alpha})}\right]-2} \rho \partial^i \partial^i \ln \rho \quad (30)$$

the equation defining the multifractal “forces” deriving from a multifractal “potential” Q can be written in the form of a multifractal equilibrium equation:

$$\rho \partial^i Q = \partial_i \hat{t}^{ii} \quad (31)$$

The multifractal tensor \hat{t}^{ii} can now be written in the form:

$$\hat{t}^{ii} = \eta (\partial_i V_F^i + \partial_i V_F^i) \quad (32)$$

with:

$$\eta = \lambda (dt)^{\left[\frac{2}{\bar{r}(\bar{\alpha})}\right]-1} \rho \quad (32)$$

This is, indeed, a multifractal linear constitutive equation for a multifractal “viscous fluid”, offering the reason for an original interpretation of coefficient η as a multifractal dynamic viscosity of the multifractal fluid. In the following, this tensor will be “made responsible” for liquid crystals’ friction type behaviors (named as the multifractal friction tensor).

SYNCHRONIZATION MODES IN LIQUID CRYSTALS DYNAMICS THROUGH A “HIDDEN” SYMMETRY OF A MULTIFRACTAL FRICTION TENSOR

The existence of the multifractal force (29) and the multifractal friction tensor (32) will be considered as the “trigger” of the liquid crystals processes leading to friction. In the one-dimensional situation, if the multifractal specific potential is constant, the following condition is satisfied:

$$\frac{\partial^2 \sqrt{\rho}}{\partial x^2} + k^2 \sqrt{\rho} = 0 \quad (34)$$

where:

$$k^2 = \frac{\varepsilon}{2\lambda^2(\mu)} \quad (35)$$

and ε is an integration constant. Then, according to a special operational procedure, a hidden symmetry of $SL(2, R)$ becomes functional. Indeed, let us rewrite the general solution of eq. (34) as:

$$\sqrt{\rho} = h e^{i(kx+\Phi)} + \bar{h} e^{-i(kx+\Phi)} \quad (36)$$

where h signifies a complex amplitude, \bar{h} denotes its complex conjugate, Φ represents a specific phase, and x is the spatial coordinate. Then, at various scale resolutions, h and Φ designate each liquid crystal entity with the general characteristic given by eq. (34), and thus the same k .

At various scale resolutions, quantities h , \bar{h} and Φ give the initial conditions, which are not the same for every liquid crystal entity; more precisely, these various entities occur in different states. The following problem is highlighted – is it possible, at various scale resolutions, to *a priori* emit a connection between h , \bar{h} and $e^{i(kx+\Phi)}$ belonging to different liquid crystals entities? The fact that eq. (36) is a solution of (34) allows an affirmative answer to this question. Indeed, at various scale resolutions, eq. (36) possesses a “hidden” symmetry expressed through the homographic group: a ratio $\sigma(x)$ of two solutions of eq. (34) is a solution of a Schwartz equation:

$$\{\sigma, x\} = \left(\frac{\sigma''}{\sigma'}\right)' - \frac{1}{2} \left(\frac{\sigma''}{\sigma'}\right)^2 = 2k^2 \quad (37)$$

where:

$$\sigma' = \frac{d\sigma}{dx}, \sigma'' = \frac{d^2\sigma}{dx^2} \quad (38)$$

and $\{\sigma, x\}$ is the Schwartzian of function σ in relation to x . Eq. (37), at various scale resolutions, is invariant to the homographic transformation of $\sigma(x)$: any such function is in itself a solution of eq. (37). Since, at various scale resolutions, homography characterizes line projectivity, it is then possible to affirm that the ratio of the two solutions of eq. (34) is a projective parameter for the totality of liquid crystal entities sharing the same k . A useful projective parameter can be constructed, which is in biunivocal correspondence with any liquid crystal entity.

First, let us observe a “universal” projective parameter – the ratio of the fundamental solutions of eq. (34) gives:

$$K = e^{2i(kx+\phi)} \quad (39)$$

Every homographic function of this type will be a projective parameter. Among others, function:

$$\sigma(x) = \frac{h+\bar{h}K}{1+K} \quad (40)$$

has the advantage to be specific to every liquid crystal entity. This is not all; let us consider another function:

$$\sigma'(x) = \frac{h'+\bar{h}'K'}{1+K'} \quad (41)$$

which is specific to a different liquid crystal entity. The fact that eqs. (40) and (41) offer solutions to eq. (37) evidences the homographic relation present between them:

$$\sigma'(x) = \frac{a\sigma(x)+b}{c\sigma(x)+d}, a, b, c, d \in \mathbb{R} \quad (42)$$

which implies, at various scale resolutions, the transformations group:

$$\begin{aligned} h' &\leftrightarrow \frac{ah+b}{ch+d'} \\ \bar{h}' &\leftrightarrow \frac{a\bar{h}+b}{c\bar{h}+d'} \\ K' &\leftrightarrow \frac{a\bar{h}+b}{ch+d} K \end{aligned} \quad (43)$$

Eq. (43), at various scale resolutions, can be assimilated to a group of synchronisms between the diverse entities of the liquid crystals, a process which

includes synchronization modes between the amplitudes and phases of liquid crystals' entities.

At various scale resolutions, the structure of this group is typical of $SL(2R)$, i.e.:

$$\begin{aligned} [B^1, B^2] &= B^1 \\ [B^2, B^3] &= B^3 \\ [B^3, B^1] &= -2B^2 \end{aligned} \quad (44)$$

where B^1 , B^2 and B^3 are the infinitesimal generators of the groups:

$$\begin{aligned} B^1 &= \frac{\partial}{\partial h} + \frac{\partial}{\partial \bar{h}} \\ B^2 &= h \frac{\partial}{\partial h} + \bar{h} \frac{\partial}{\partial \bar{h}} \\ B^3 &= h^2 \frac{\partial}{\partial h} + \bar{h}^2 \frac{\partial}{\partial \bar{h}} + (h - \bar{h})K \frac{\partial}{\partial K} \end{aligned} \quad (45)$$

This group admits the absolute invariant differentials:

$$\begin{aligned} \omega^1 &= \frac{dh}{(h - \bar{h})K} \\ \omega^2 &= i \left(\frac{dK}{K} - \frac{dh + d\bar{h}}{h - \bar{h}} \right) \\ \omega^3 &= -\frac{K d\bar{h}}{h - \bar{h}} \end{aligned} \quad (46)$$

and the 2-form (the metric):

$$ds^2 = \left(\frac{dK}{K} - \frac{dh + d\bar{h}}{h - \bar{h}} \right)^2 - \frac{dh d\bar{h}}{(h - \bar{h})^2} \quad (47)$$

In real terms:

$$h = u + iv, \bar{h} = u - iv, K = e^{i\theta} \quad (48)$$

and for:

$$\begin{aligned} \Omega^1 &= \omega^2 = d\theta + \frac{du}{v} \\ \Omega^2 &= \cos \theta \frac{du}{v} + \sin \theta \frac{dv}{v} \\ \Omega^3 &= \cos \theta \frac{dv}{v} - \sin \theta \frac{du}{v} \end{aligned} \quad (49)$$

the connection with a Poincaré representation of the Lobachevsky plane at various scale resolutions can be obtained. Indeed, at various scale resolutions, the metric is a three-dimensional Lorentz structure:

$$ds^2 = -(\Omega^1)^2 + (\Omega^2)^2 + (\Omega^3)^2 = -\left(d\theta + \frac{du}{v}\right)^2 + \frac{du^2 + dv^2}{v^2} \quad (50)$$

This metric reduces to that of Poincaré in case where $\Omega^1 \equiv 0$, which defines variable θ as the “angle of parallelism” of the hyperbolic planes (the connections). In fact, recalling that:

$$\frac{dK}{K} - \frac{dh + d\bar{h}}{h - \bar{h}} = 0 \leftrightarrow d\theta = -\frac{du}{v} \quad (51)$$

represents, at various scale resolutions, the connection form of the hyperbolic plane, eq. (50) then represents a general Bäcklund transformation in that plane. In such a conjecture, the metric represented by eq. (47) with the restrictions represented by (51) becomes:

$$ds^2 = \frac{dh d\bar{h}}{(h - \bar{h})^2} = \frac{du^2 + dv^2}{v^2} \quad (52)$$

Worth mentioning is that, at various scale resolutions, a property connected to integral geometry: the group in eq. (43), is measurable [40–42]. Indeed, it is simply transitive, since its structure vectors:

$$C_\alpha = C_{v\alpha}{}^v \quad (53)$$

are identically null, as seen from eq. (44). This means that, at various scale resolutions, it possesses an invariant function given by:

$$F(h, \bar{h}, K) = -\frac{1}{K(h - \bar{h})^2} \quad (54)$$

As a result, at various scale resolutions, in the space of variables (h, \bar{h}, K) , a probability theory can *a priori* be constructed, based on the elementary probability:

$$dP = -\frac{dh \wedge d\bar{h} \wedge dK}{K(h - \bar{h})^2} \quad (55)$$

where \wedge denotes the exterior product of the 1-forms. This agrees with Jaynes’s theorem, wherein it can be implied that, at any scale resolutions, any circumstance left unspecified regarding a certain dynamic in an experiment can find its expression in the group $SL(2, R)$. This means that the given circumstance is not left unspecified for subjective reasons, but for the fact that the particulars of the given dynamics are not manifested.

MULTIFRACTAL FRICTION BEHAVIORS IN LIQUID CRYSTALS DYNAMICS THROUGH HARMONIC MAPPINGS

Let us assume that the liquid crystals dynamics is described, at various scale resolutions, by variables (Y^j) , for which we have discovered the metric:

$$h_{ij} dY^i dY^j \quad (56)$$

in an ambient space of metric:

$$\gamma_{\alpha\beta} dX^\alpha dX^\beta \quad (57)$$

The field equations of liquid crystal dynamics are obtained from a variational principle, which, in this case, is linked to the Lagrangian, at various scale resolutions:

$$L = \gamma^{\alpha\beta} h_{ij} \frac{\partial Y^i \partial Y^j}{\partial X^\alpha \partial X^\beta} \quad (58)$$

In our scenario, eq. (56) is equal to eq. (52), h and \bar{h} being the field variables or, equivalently, the real and imaginary part of h . As a result, if the variational principle is applied at various scale resolutions:

$$\delta \int L \sqrt{\gamma} d^3 X \quad (59)$$

it is accepted as a starting point, where $\gamma = |\gamma_{\alpha\beta}|$, the main goal of the liquid crystals friction type dynamics analysis would be to produce metrics of the Lovachevski plane (or relate to them). In this case, the Euler equations corresponding to the variational principle (59) are:

$$(h - \bar{h}) \nabla(\nabla \bar{h}) = 2(\nabla \bar{h})^2 \quad (60)$$

which admit the solution:

$$h = \frac{\cosh(\Phi/2) - \sinh(\Phi/2)e^{-i\alpha}}{\cosh(\Phi/2) + \sinh(\Phi/2)e^{-i\alpha}}, \alpha \in \mathbb{R} \quad (61)$$

with α real and arbitrary, as long as $(\Phi/2)$ is the solution of a Laplace-type equation for the free space, so that $\nabla^2(\Phi/2) = 0$. For the choice of form $\alpha = 2\Omega t$, when a temporal dependency is included into the liquid crystal dynamics at various scale resolutions, eq. (61) becomes:

$$h = \frac{1 + ie^{\Phi} \tan(\Omega t)}{e^{\Phi} + i \tan(\Omega t)} \quad (62)$$

Figures 2–4 plot the various cellular friction type behaviors of liquid crystals dynamics at different scale resolutions in dimensionless coordinates: i) at global scale resolution (Fig. 2a,b); ii) at differentiable scale resolution (Fig. 3a,b); iii) at non-differentiable scale resolution (Fig. 4a,b). Let us observe that, regardless of scale resolution, the dynamics of liquid crystals may be reduced to self-structuring cellular friction patterns.

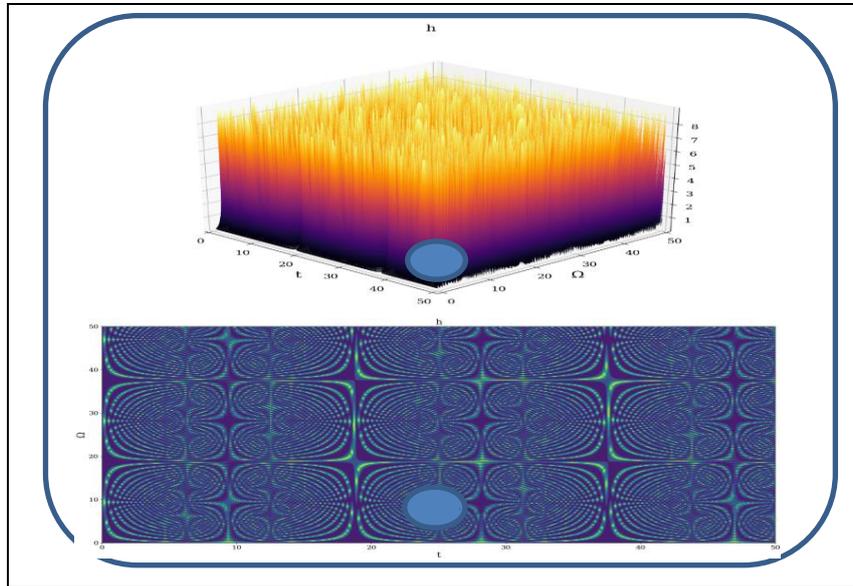


Figure 2. Friction type cellular patterns at global scale resolution: 3D plot of $h(\Omega, t)$; $\Phi = 2.35$ (a); 2D plot of $h(\Omega, t)$; $\Phi = 2.35$ (b).

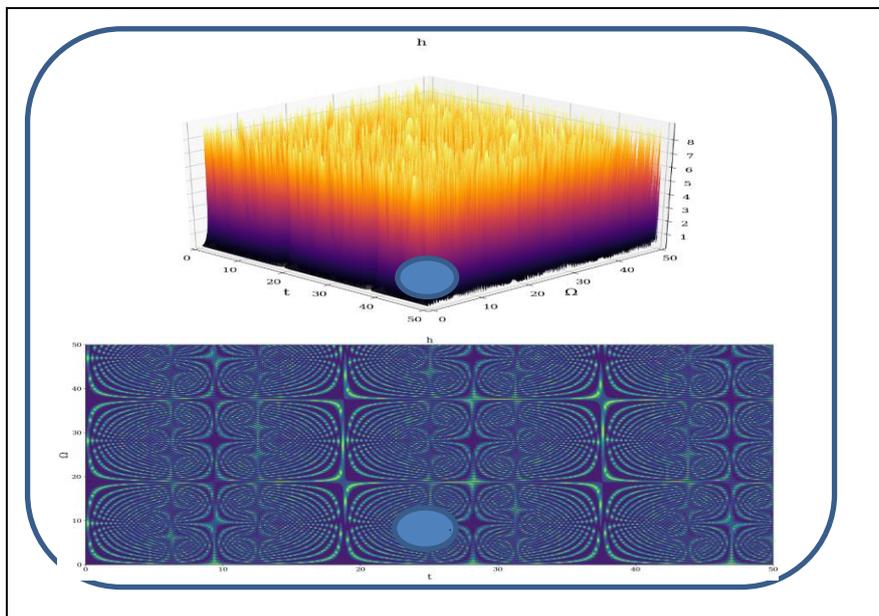


Figure 3. Friction type cellular patterns at differentiable scale resolution: 3D plot of $Re[h(\Omega, t)]$; $\Phi = 2.35$ (a); 2D plot of $Re[h(\Omega, t)]$; $\Phi = 2.35$ (b).

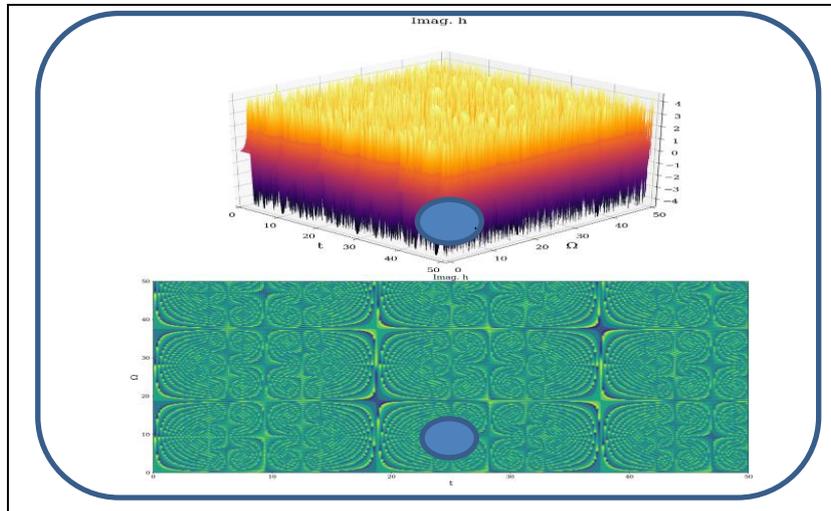


Figure 4. Friction type cellular patterns at non-differentiable scale resolution: 3D plot of $Im[h(\Omega, t)]$; $\Phi = 2.35$ (a); 2D plot of $Im[h(\Omega, t)]$; $\Phi = 2.35$ (b).

Let us also observe that the Multifractal Theory of Motion's mathematical formalism naturally implies a variety of operational techniques (invariance groups, harmonic mappings, groups isomorphism, embedding manifolds, and so on) with numerous applications in liquid crystals dynamics. When h is plotted in dimensionless parameters, some temporal self-similar patterns emerge, with multifractal structures contained in similar multifractal structures at even larger scales (Figs. 5 a-c). This behavior is quite difficult to represent, because of the complicated balance between choosing an adequate number of plot points manifesting self-similarity, but it shows how the small-scale behavior of the system ripples and manifests itself at higher scales, which is exactly what we would expect from a multifractal system.

The results shown in Figure 5 also show that liquid crystal entities can be self-structuralized, and that the scale transitions induced through the modification of fractal dimensions of the liquid crystal dynamics movement curves can be assimilated to friction type channels.

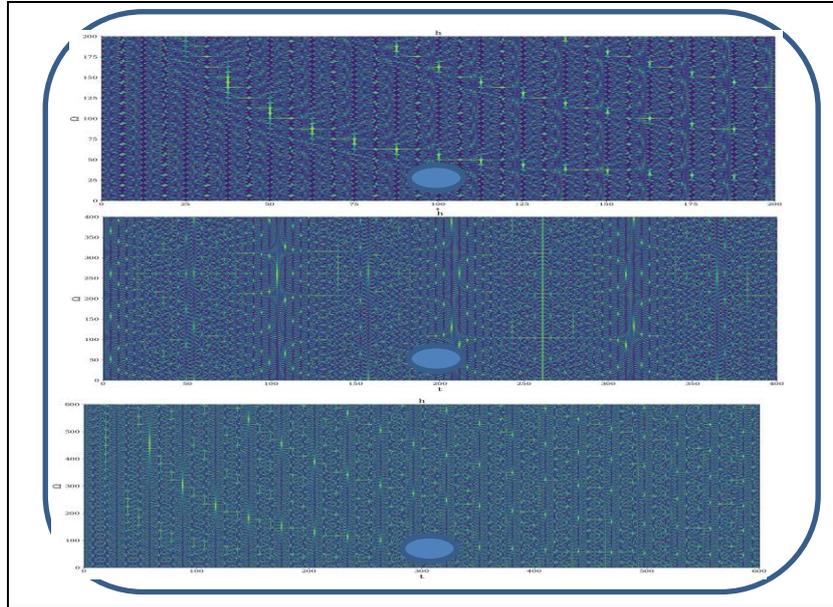


Figure 5. Friction type channels for various fractal degrees: 2D plot of $h(\Omega, t)$, maximum at 200; $\Phi = 2.35$ (a); 2D plot of $h(\Omega, t)$, maximum at 400; $\Phi = 2.35$ (b); 2D plot of $h(\Omega, t)$, maximum at 600; $\Phi = 2.35$ (c).

CONCLUSIONS

The main conclusions of this paper are the following:

Assuming that any liquid crystal may be assimilated to a mathematical entity of multifractal type, various non-linear friction behaviors are highlighted in relation with the Multifractal Theory of Motion. In this context, Schrödinger's multifractal motion curves and Madelung's multifractal description of any liquid crystal entity dynamics become operative.

A hidden symmetry of a multifractal friction tensor is investigated at various scale resolutions in the Madelung framework, the synchronization modes among the liquid crystals entities being found through $SL(2R)$ transformations groups.

The amplitudes and phases of motions of any liquid crystal entities are related during synchronization, while the amplitude attributed to each motion is also impacted from a multifractal homographic perspective. The typical synchronization modes used here, appearing as a delay in the amplitude and phases of the liquid crystal entities, are only a particular case.

At various scale resolutions, the space induced by $SL(2R)$ group parameters can be formed as Riemann manifolds (multifractal Riemann manifold). The

generators and its associated metric were found. When a direction parallel transport in the Levi-Civita sense became functional, the metric can be reduced to that of Poincaré, with the angle of parallelism of the hyperbolic plane defining the connections.

The direction parallel transport in the Levi-Civita sense can be correlated with the apolar transport of cubics generated through the multifractal friction tensor eigenvalues.

In this framework, access to non-stationary dynamics at various scale resolutions becomes possible *via* harmonic mapping from the usual space to the hyperbolic one. Then, through self-structuring (induced by the synchronization modes among the liquid crystal entities), non-linear friction-type behaviors in the form of cellular and channel patterns are manifesting.

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