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THE INFINITE IS THE CHASM IN WHICH OUR THOUGHTS ARE LOST: REFLECTIONS ON SOPHIE GERMAIN'S ESSAYS

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Sophie Germain (1776–1831) is quite well-known to the mathematical community for her contributions to number theory [17] and elasticity theory (*e.g.*, see [2,5]). On the other hand, there have been few attempts to understand Sophie Germain as an intellectual of her time, as an independent thinker outside of academia, and as a female mathematician in France, facing the prejudice of the time of the First Empire and of the Bourbon Restoration, while pursuing her thoughts and interests and writing on them. Sophie Germain had to face a double challenge: the mathematical difficulty of the problems she approached and the socio-cultural context of her time, which never fully supported her interests, never appropriately rewarded her, and never allowed her to enjoy the recognition she deserved. In our attempt to understand the innermost Sophie Germain, we also try to grasp the place of her personality within her time and historical period. We will argue that she represents a unique case in both the history of mathematics and the context of Western European intellectuals at the beginning of the 19th century, deserving a further exploratory study on the connections of her work with the ideas of her time.

Deservingly, toward the end of the twentieth century, Sophie Germain's works received attention in several thorough and useful inquiries, [*e.g.*, 13–15]. However, a specialist during this same period deemed her work as not worthy of glory [18], and she was even described as a "minor author" [19]. This is why we posit that further analysis and careful discussion of her intellectual achievements – mathematical or otherwise – is necessary. Our goal is to better assess her important contributions,

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and to invite the consideration of her achievements and vision in the same manner as the ones of her contemporaries, such as Gauss, Lagrange, Cauchy, and Poncelet.

We will start our argument with the uncontroversial fact that Sophie Germain is the mathematician who first introduced the concept of mean curvature [9]. This concept is a fundamental one in differential geometry [3] and its introduction generated a profound discussion about minimality in the geometry of submanifolds that is still relevant today, and that led to the study of a plethora of new curvature invariants [3]. This turning point in differential geometry led to, among thousands of other results, the recent investigations on Willmore energy, which in turn brought us fundamental new results in differential geometry, such as the solution of the Wilmore Conjecture by Marques and Neves [12]. Sophie's work also became a historical starting point for the 2018 Abel Prize winner Karen Uhlenbeck or her work in geometric analysis [20]. We thus propose that Sophie Germain's introduction of *la courbure moyenne* [9] defines her as a mathematician deserving of the highest attention for her mathematical vision and of the most profound recognition for her intellectual standing.

Sophie Germain was a trailblazer both as a female mathematician and as a differential geometer introducing an important invariant used and generalized in today's geometrical theories. Her influence eased the way for other mathematical giants, such as Emmy Noether, or later, Michelle Audin, Dusa Mc Duff, Chuu-Lian Terng, as well as the 2014 Fields Prize winner Maryam Mirzakhani. Following in her historical footsteps, we can see how these wonders of mathematics became inspiration for future generations of female mathematicians, may they be differential geometers, algebraists, or topologists.

To better assess the complexity of Sophie Germain's body of work in the context of her contributions to mathematical history as a woman mathematician, we should compare it with those of other important cultural giants who played a singular part in their respective historical period. One such comparison could be made with Christine de Pizan (1364–1430), one of the first professional writers in medieval Europe, a biographer of King Charles V, and a first-hand witness of a historical period, when her contemporary society descended into chaos and war. Barbero describes Christine de Pizan [1] as "the first feminist" and "*une femme engagée*", *i.e*., an independent intellectual who acts according to her principles and convictions while responding to the challenges of her time. If Christine de Pizan definitely is such an intellectual, exactly in the terms described by Barbero, then we should discuss Sophie Germain's imprint on the history of her times along the same lines, taking into account all aspects of her historical environment, from social prejudice, to her life in the time of war and social tensions that would lead to permanent changes to French society. As in the case of Christine de Pizan, whose principles determined an attitude that today would be described as political, Sophie Germain did not hesitate to act in support of her values. For example, she did everything in her power, using all of her political influence, to protect Gauss when the French imperial army invaded his hometown (see her correspondence with

General Pernety, who, in 1806, directed the siege of Breslau [10], pp. 316–317).

Jane Austen (1775–1817) was another contemporary of Sophie Germain. Similar to Sophie, Jane Austen published anonymously, her name not appearing on her works until after her death. Her writing style presaged the literary realism movement, and the themes and political observations in her writing were so nuanced and important as to have legitimate claims by both conservatives and liberals. After comparing her appeal to Shakespeare and Dickens, Austen scholar John Mullan ([16], p. 2) writes that "...she did things with fiction that had never been done before. She did things with characterization, with dialogue, with English sentences, that had never been done before." It is not surprising, then, that there are hundreds of works of literary criticism devoted to Jane Austen's writings. We can only hope that Sophie's work will receive the same type of interest as Jane's literary and intellectual contributions.

While her mathematics was astounding, the scope of Sophie's intellectual brilliance is much wider. To better support our interest in all facets of her personality, we cite her volume of *Philosophical Works*, published in 1879 by Paul Ritti [10]. In particular, within this volume we refer the interested reader to a longer essay titled *General considerations on the state of sciences and of the letters in different times of their cultures*, a series of short essays titled *Pensées*, as well as previously unreleased letters. Important information on her private correspondence was investigated and published only recently by A. Del Centina [6,8], doing justice to such an interesting intellectual giant. All of these elements and texts should be taken into consideration when one discusses Sophie Germain's intellectual span and vision, and we feel that her intellectual life should be as important to the mathematical community as Jane Austen's intellectual vision is to the literary community.

Despite the above mentioned discussions of her philosophical works, we feel there is more to be done. Here, we briefly describe the main text in the volume titled by the editor *Philosophical Works*, *i.e*. the long essay *General considerations on the state of sciences and of the letters in different times of their cultures*.

In the first chapter, Sophie Germain argues that, in various cultures, the development of sciences and the evolution of letters (including poetry and fiction) are governed by a common spirit while, in the second chapter, she starts by remarking that literature appeared in all world cultures before science. Her inquiry is not mathematical, and it definitely pertains to the philosophy of culture, as Sophie Germain is much interested in the origins of scientific inquiry, and this is best described in the following paragraph:

> "*Les sciences n'existaient pas encore; mais le besoin d'expliquer s'était fair sentir. La première des littératures fut poétique. Ce qui tenait lieu des sciences physiques n'était pas moins poétique que la littérature elle-même ou plutôt ces deux branches du savoir, tellement séparees aujourd'hui qu'il faut de l'attention pour remarquer ce qu'elles ont du commun, étaient dans ces premiers temps entièrement confondues.*"

> ("The sciences did not exist yet, but the need to explain was beginning to be felt. The first of the literatures was poetic. What took place in physical science was no less poetic than in literature itself, rather the two branches of knowledge, so far separate today that much attention is needed to identify what they have in common, were originally entirely entangled.") ([10] p. 113)

This transdisciplinary remark reveals not just the reflection of a research mathematician at work in the first decades of the 19th century, but a thorough historical vision. While it may be that her particular considerations are a product of the spirit of her time, it is important to point out that, by transcending the limitations of a single area of knowledge, most of Sophie Germain's essays exceed the vision and depth of most working mathematicians' reflections.

By the very fact that this mathematician, with important and numerous contributions to number theory, elasticity theory and differential geometry, ventures into the territory of the philosophy of culture, we recognize that Sophie Germain is an authentic intellectual of her time, with a manifold interest in a variety of challenging ideas, who follows closely not only the current vents and affairs of her era (*e.g.*, the developments of the Napoleonic wars), but also their historical causalities. Although prejudices against females in academia prevented her from participating formally, she was very familiar with contemporary schools of thought and had a sophisticated perspective on the role of science in society. We also see that Sophie Germain had well-shaped opinions on a variety of scientists and their very specific work, as she also describes her preferences amongst them; she refers in her essay to a series of authors, some classics, such as Descartes and Newton, and some of her contemporaries, such as Immanuel Kant. She notes that:

> "*Newton parut, armé d'un nouveau genre de calcul: et l'unité, l'ordre, les proportions de l'univers que le sentiment du vrai avait fait chercher si longtemps devinrent des vérités mathématiques. Son génie avait reconnu la cause des mouvements célestes: une analyse pleine de finesse lui servit à les mesurer.*"

("Newton appeared, armed with a new kind of calculus: and the unity, order, and proportions of the universe whose true reality had long been searched became mathematical truths. His genius recognized the cause of celestial movements: an analysis full of refinement served him to measure them.") ([10], p. 146.)

The unity of concepts was a modernist thought, which became highly valued a century later, and Sophie Germain points it out several times throughout her historical reflections and she seems to find, in Isaac Newton, a moral model and a more general example to follow. She notes:

> "*En parlant de Newton qui fut solitaire et modeste, qui ne chercha point à paraître, qui fit des grandes choses avec simplicité, il faut être simple comme lui, comme la nature qu'il a suivie. Cette simplicité qui le charactérise est la grandeur que son écrivan doit emprunter de lui.*"

("Speaking of Newton, who was a loner and modest being, who did not seek to show off, who did great things with simplicity, one must be simple like him, like the nature he has followed. This simplicity which characterizes him is the greatness that any writer must borrow from him.") ([10], p. 258.)

Sophie Germain also reflects on what real life actually reserved for mathematicians during and before her time, and these reflections are as relevant today; one can feel in her ethical quest a reflection on her own destiny as mathematician. She writes about others, but in many ways she writes about herself when she says:

"*Tycho [Brahe] avait été destiné à la jurisprudence, comme Copernic le fut à la médecine.*" ("Tycho was as destined to the legal profession, as Copernicus was to medicine.") ([10], p. 243.)

At some point she seems to criticize Tycho Brahe for his lack of philosophical reflection ([10] p. 247, p. 255), but she finds his attitude understandable, as he was a man much influenced by his century, where the interests in alchemy merged with astronomical observations. By comparison, Sophie Germain has a much more positive take on Dominique Cassini's works and heritage ([10] pp. 256–257) whose works she finds "*précieux*."

Sophie Germain is ultimately interested in what she sees as the fundamental duty of being a mathematician. Reminiscent of Hardy's *Apology* [11], she writes this reflection on the proper definition of a geometer:

"*Un géométre est un homme qui entreprend de trouver la vérité, et cette recherche est toujours pénible dans les sciences comme dans la morale. Profondeur de vue, justesse de jugement, imagination vive, voilá les qualités du géométre. Profondeur de vue pour apercevoir toutes les conséquences d'un principe, cette immense postérité d'un même pére. Justesse de jugement, pour distinguer entre elles les traits de famille, et pour remonter de ces conséquences isolées au principe dont elles dépendent. Mais ce qui donne cette profondeur, ce qui exerce ce jugement, c'est l'imagination, non celle qui se joue á la surface des choses, qui les anime de ses couleurs, qui y répand l'éclat, la vie et le mouvement, mais une imagination qui agit au dedans des corps comme celle-ci au dehors. Elle se peint leur constitution intime, elle la change et la dépouille á volonté; elle fait, pour ainsi dire, l'anatomie des choses et ne leur laisse que les organes des effets qu'elle veut expliquer. L'une accumule pour embellir, l'autre divise pour connaître. L'imagination qui pénétre ainsi la nature; vaut bien celle qui tente de la parer. Moins brillante que l'enchanteresse qui nous amuse, elle a autant de puissance et plus de fidélité. Quand l'imagination a tout montré, les difficultés et les moyens, le géométre peut aller en avant; et s'il est parti d'un principe incontestable, qui rende sa solution certaine, on lui reconnaît un esprit sage. Ce principe le plus simple offre-t-il la voie la plus courte, il a l'élégance de son art. Et enfin il a du génie, s'il atteint une vérité grande, utile et longtemps déparée des vérités connues.*"

("A geometer is a man who undertakes to find the truth, and this research is always as painful in science as in morality. Depth of sight, correctness of judgment, lively imagination, these are the qualities of the geometer. Depth of sight to see all the consequences of a principle, this immense posterity from the same father. Judgment correctness, to distinguish the family traits between them, and to go back from these isolated consequences to the principle from which they spring. But what gives this depth, which exercises this judgment, is the imagination, not what is played on the surface of things, which animates them with its colors, which diffuses brightness, life, and movement, but an imagination that works just as well inside bodies as it does outside. It paints their intimate constitution, it changes it and strips it at will; it describes, so to speak, the anatomy of things, and leaves them only the organs of the effects which it wishes to explain. One accumulates for embellishment, the other divides for knowledge. The imagination thus penetrates nature; the one who tries to parry it is worthy. Less brilliant than the enchantress who amuses us, she has as much power and more fidelity. When the imagination has shown everything, the difficulties and the means, the geometer can go forward; and if he has started from an incontestable principle, which renders certainty to his solution, he is recognized as having a wise mind. This simplest principle offers the shortest route, it has the elegance of its art. And finally he has genius, if he proves a great truth, useful, and far removed from known truths.") ([10] pp. 266–267.)

Furthermore, any geometer would quantify the following fragment as one of the most interesting in her works, as it is premonitory and substantive in every sense:

La géométrie est la science de l'étendue et du mouvement ou seulement de l'étendue: car tout ce qui existe dans cet univers, ou á la fois ou successivement, a l'étendue pour caractére de son existence. L'espace qui embrasse tous les points, tous les lieux, toutes les bornes du physique; le mouvement qui parcourt cet espace, qui s'y applique, s'y mesure et semble s'y assimiler; le temps marqué par la succession des choses, subsistant depuis leur commencement jusqu'á leur fin; le temps qui embrasse l'univers dans ses changements, comme l'espace l'enferme dans sa permanence, tout n'est qu'étendue. Etendue physique qui est devant nous, que l'œil peut distinguer et parcourir, étendue intellectuelle que l'homme peut rendre présente á son esprit et qui n'est aperçue et mesurée que par la pensée. Voilá l'empire de la géométrie. C'est alors qu'elle est grande, qu'elle est vaste comme l'univers! Ouvrage miraculeuse de la raison humaine, les hommes y ont concentrée toutes les idées d'ordre et de rectitude, qu'ils ont reçues du ciel."

("Geometry is the science of magnitude and of movement, or only of magnitude, since all there is in this universe, either simultaneously or successively, has magnitude to characterize its existence. The space which embraces all points, all places, all boundaries of the physical world, the movement that passes through this space, which applies here, which is measured here, and is assimilated here; the time marked by the succession of events, existing from their beginning up to their end; the time which embraces

> the universe in its changes, as well as the space confined in its eternity, all is nothing but magnitude. It is the physical magnitude that lies ahead, that our eyes can distinguish and cover, intellectual magnitude that the man can spark in his spirit and which cannot be perceived and measured by anything else but by thought. That is geometry's empire. That is how large it stands, as wide as the whole universe! A wonderful miracle of human reason, people have focused inside all the ideas on order and on straightness they have received from the heavens.") ([10] pp. 262–263.)

Sophie Germain's interests pursued the fundamental principle to the ultimate realm, where she inventively resorts to effective metaphors to make her point:

> "*La nature n'est que mélange et tempéraments, deux principes destructeurs l'un par l'autre enchaînés sont unis pour des effets durables. L'alliance de ces principes maintient la société des corps célestes! Rien n'est plus admirable que ce mécanisme, c'est par cette combinaison de forces que tout se meut, tout change et cependant tout se conserve!*"
>
> ("Nature is nothing else but mixture and disposition. Two principles destroying one another are interconnected to yield long lasting consequences. The alliance of these principles keeps the combination of celestial bodies! Nothing is more admirable than this mechanism, due to this combination of forces that everything moves, everything changes, and in the same time everything is conserved.") ([10] pp. 258–259)

We would be remiss to forget Sophie Germain's note on human nature at a time when the Napoleonic wars left Europe devastated:

> "*Nos moyens pour surpasser la science primitive ont donc été le télescope qui étend le domaine des sens, la géométrie qui permet de tout approfondir et le génie qui ose tout comparer et qui s'élève à la science des causes. Cette science est notre véritable supériorité. Tous les phénomènes sont enchaînés. Le système de nos connaissances est ordonné comme la nature; un seul principe nous sert à tout expliquer, comme un seul effort lui suffit pour faire tout agir.*"
>
> ("Our means to exceed primitive science have been the telescope, which extends the domain of the senses, geometry, which allows us to deepen everything and the genius which dares to compare everything and which elevates to the science of causality. This science is our true superiority. All the phenomena are entangled. The system of our knowledge is as ordered as nature; one single principle serves us to explain everything, as a single effort is enough [to this principle] to make everything happen.") ([10] p. 281)

This paragraph is strongly reminiscent of Leo Tolstoy's concluding remarks from *War and Peace*, where the novelist is looking for the ultimate principles that govern major events such as the Napoleonic Wars. Tolstoy's masterpiece was

published in its entirety in 1869 and this is how history felt in the 19th century. Consequently, we contend that Sophie Germain should be viewed not only as a research mathematician, but as a deep thinker, an intellectual facing and reflecting upon her time and on the forces of the natural world.

In the end, we are convinced that Sophie Germain feels most at home when she comments on mathematics and we embrace her clear vision on relationships and entanglements between various chapters and concepts of mathematics. In this vein, Sophie Germain anticipates:

> "*La méthode complète du calcul intégral serait une révolution dans la géométrie semblable à celle de l'application de l'algèbre et à celle de l'invention du calcul différentiel.*"
>
> ("The complete method of the integral calculus would be a revolution in geometry similar to that of the applications of algebra and of the differential calculus.") ([10] p. 281)

If we take into account her overall writings, her essays and her private correspondence, Sophie Germain reveals herself as a fascinating scientist with an interesting humanistic personality, possessing eclectic interests, a very complex vision of mathematics and of the role of science in the world, as well as a personal vision of culture and philosophy, revealed in her vast array of reflections, composed in a unique and exquisite style. We can only speculate and wonder at what accomplishments such an active and brilliant mind would have achieved if Sophie Germain had been allowed to pursue her interests to their highest academic potential. In her destiny, there exists a historical lesson for us all. The history of mathematics simply does not have any other case of a researcher with such subtle and fundamental contributions, who faced a similar comprehensive system of prejudices and barriers, and who left such a transdisciplinary heritage. The historians of science, the translators, and the mathematicians who investigate her work perform a great service to the mathematical community.

We would like to end this well deserved panegyric with Sophie Germain's own words. She writes the following in a poetical note that can only be described as a mark of her personal style for this entire genuine diary of ideas:

> "*L'infini est le gouffre où se perdent nos pensées; il n'est pas naturel de se jeter dans des précipices. Si l'homme est descendu dans cet abîme sans fond, il y fut entraîné par une pente.*"
>
> ("The infinity is the chasm in which our thoughts are lost; it's not natural to throw oneself in its precipices. If the man descends in this endless abyss, he would be dragged into a fall.") ([10] p. 235.)

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