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MATHEMATICS

GENERALIZED SKEW-SEMI INVARIANT SUBMANIFOLDS OF ALMOST PARACONTACT MANIFOLDS

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We define and study here a special class of submanifolds in almost paracontact Riemannian manifolds which correspond to the generic submanifolds in the Kæhlerian case and in the almost contact case. Therefore, we fill a gap in literature. An example is constructed at the end.

Key words: almost paracontact manifold, Riemannian submanifold

1. INTRODUCTION

Almost paracontact geometry appeared as a counterpart of almost contact geometry, as well as the almost contact geometry (see [4]) emerged as the odd dimensional version of complex geometry. As explained in [1], there are two directions on which the almost paracontact geometry has developed. The first one was initiated by Sato in [10], the other was introduced later on by Kaneyuki-Kozai in [6] and Kaneyuki-Williams in [7]. In our paper, we shall follow the first direction, namely the Riemannian metric compatible with the almost paracontact structure. The importance of almost paracontact manifolds consists, on one side, of extending the almost para-Hermitian structures and, on the other side, of modeling some physical phenomena.

The aim of the present note is to give and study the correspondent notion in the almost paracontact context for the generic submanifolds in the Kæhlerian case (see [9]) and in the almost contact case (see [2]). We study here the behaviour of these submanifolds, and in the end we construct an example.

We shall use the following notion of almost paracontact manifold, introduced for the first time in the literature by Sato [10,11] which is slightly different from the one given by Kaneyuki-Kozai [6] and Kaneyuki-Williams [7].

Definition 1. A n-dimensional manifold M is called almost paracontact if there is given a triple (F, ξ, η), named paracontact structure, of a (1,1)-tensor field F (called structure tensor field), a vector field ξ (called Reeb vector field) and a one-form η (called paracontact form), globally defined on M, so that:

$$F^2 = I - \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1 \tag{1}$$

Let D^* and D^- denote the eigen distributions corresponding, respectively, to the eigenvalues (+1) and (-1) of F.

Remark 2. **(i)** $F\xi = 0$; **(ii)** $\eta \circ F = 0$; **(iii)** rank $F = n-1$ but n may be either odd or even; **(iv)** $\mathrm{Ker}\eta = \mathrm{Im}F$ **(v)** $\mathrm{Ker}F = span\{\xi\}$; **(vi)** F restricted to distribution $\mathrm{Ker}\eta$ is a product structure whose eigen distributions D^+ and D^- may have different ranks; **(vii)** any paracontact manifold (M, F, ξ, η) admits a positive definite Riemannian metric g such that:

$$g(FX, FY) = g(X,Y) - \eta(X)\eta(Y), \forall X, Y \in \Gamma(TM). \tag{2}$$

By using (1) and (2), one can obtain:

$$\eta(X) = g(X, \xi), \forall X \in \Gamma(TM). \tag{3}$$

Let $D = \mathrm{Im}F = Ker\eta$ denote the paracontact distribution of the manifold M. Hence, the tangent bundle decomposes into the direct orthogonal sum:

$$TM = D \oplus span\{\xi\}. \tag{4}$$

Definition 3. [11] A Riemannian manifold (M, g) endowed with an almost paracontact structure (F, ξ, η) satisfying (2) is called an almost paracontact Riemannian manifold.

Remark 4. We note that, in recent times, under the name of almost paracontact metric manifold, a slightly different definition is provided (see [3]). Moreover, inspired by the original definition given by Sato ([10,11]), some other related notions were introduced in literature (see, for instance [5]).

2. SOME SPECIAL (1,1)-TENSOR FIELDS ON SUBMANIFOLDS

This section contains some fundamental operators to be used later on.

Let (M, F, ξ, η, g) be an almost paracontact Riemannian manifold and let N be a submanifold of M . For any $X \in \Gamma(TN)$, we may write:

$$FX = PX + QX \tag{5}$$

where $PX \in \Gamma(TN)$ and $QX \in \Gamma((TN)^{\perp})$

From (1) and (2), there follows that F is symmetric. By using (1), (2), (5) and the fact that almost paracontact structure F is symmetric, we obtain the following:

Proposition 5. Let N be a submanifold of an almost paraontact Riemannian manifold (M,F,ξ,η,g) and let P the operator defined by (5). Then:

1. P and P^2 are symmetric operators with respect to g on N;

2. all eigenvalues of P^2 are contained in [0,1].

Remark 6.

- From the above proposition, P^2 has at each point the associated matrix diagonalizable;
- From now on, any eigenvalue of P^2 will be denoted by ν^2, $\nu \in [0,1]$;
- We may write $\nu = \cos\alpha(u)\sin\theta(u), \alpha(u) \in [0,\pi], \theta(u) \in [0,\frac{\pi}{2}]$ where $\alpha(u)$ denotes the angle between ξ and u and $\theta(u)$ the angle between Fu and Pu, for any unitary eigenvector u of P^2;
- The existence of the Reeb vector field shows that our work (in almost paracontact geometry), is different from [9,13] in Kæhlerian geometry, from [2] in almost contact geometry and from almost product geometry [8,12];
- Different from the almost complex case and the almost contact case, in the almost paracontact framework, the (1,1)-tensor P is symmetric.

Corollary 7. If N is a submanifold of an almost paracontact manifold (M,F,ξ,η,g), then the following conditions are equivalent:

- N is a leaf of the contact distribution of M;
- operator P coincides with the restriction of F to N;
- operator Q is identically zero;
- the only eigenvalue of P^2 is 1;
- the above angle $\alpha(u) = 0$, for any u unitary eigenvector of P^2;
- ξ is orthogonal to N at any point $p \in N$.

3. GENERALIZED SKEW SEMI-INVARIANT SUBMANIFOLDS

In this section, we shall introduce the main notion of our paper.

Let ν^2 be an eigenvalue of P^2 whose corresponding eigen distribution will be denoted by Δ^ν. Since P^2 is diagonalizable, we may take $\nu_1^2(p),...,\nu_n^2(p)$ to be all distinct eigenvalues of P^2 at any point $p \in N.$, which yields the decomposition of T_pN into the direct orthogonal sum, *i.e.*:

$$T_pN = \Delta_p^{\nu_1} \oplus ... \oplus \Delta_p^{\nu n}. \tag{6}$$

Definition 8. A submanifold N of an almost paracontact manifold (M,F,ξ,η,g) is called a generalized skew semi-invariant submanifold if there exist some functions $\lambda_1,...,\lambda_k : N \to (0,1)$, for a positive integer k, such that at each $p \in N.$:

- $\lambda_1,...,\lambda_k$ are distinct eigenvalues of P^2;
- the dimension of each $\Delta_p^0,\Delta_p^1,\Delta_p^{\lambda_1},...,\Delta_p^{\lambda_k}$ is independent on $p \in N.$, where Δ_p^{λ} denotes the eigenspace corresponding to the eigenvalue λ(p) of P^2, for $\lambda \in \{0,1,\lambda_{1,...,\lambda_k}\}$;
- the tangent space decomposes into the direct orthogonal sum

$$TN = \Delta_p^0 \oplus \Delta_p^1 \oplus \Delta_p^{\lambda_1} \oplus ... \oplus \Delta_p^{\lambda_k}.$$

This class of submanifolds is called generalized skew semi-invariant because the slant angle of the distributions is non-constant and different from Liu's paper [8], where the submanifolds are called skew semi-invariant since the angle is constant.

Proposition 9. If N is a generalized skew semi-invariant submanifold of an almost paracontact manifold (M,F,ξ,η,g), then any distribution Δ^{ν_i} is $P-$invariant, for $i \in \{1,...,n\}$.

Proof. Let us fix an arbitrary $i \in \{1,...,n\}$ and let ν_i^2 be an eigenvalue of P^2 whose associated eigen distribution is Δ^{ν_i}. For any $j \in \{1,...,n\}, j \neq i,$ the symmetry of P yields:

$$\nu_i^2 g(PU,V)=\nu_i^2 g(U,PV)=g(P^2U,PV)=g(PU,P^2V)=\nu_j^2 g(PU,V),$$
$$\forall U \in \Gamma(\Delta^{\nu_i}), \forall V \in \Gamma(\Delta^{\nu_j})$$

Since the two eigenvalues are distinct, there follows:

$$g(PU,V)=0, \forall U \in \Gamma(\Delta^{\nu_i}), \forall V \in \Gamma(\Delta^{\nu_j})$$

which completes the proof.

By using (2), (5) and the symmetry of P, there follows:

Lemma 10. If N is a submanifold of an almost paracontact manifold (M,F,ξ,η,g), then under the above notations, we have:

$$\|QX\|^2 = \|X\|^2 - (\eta(X))^2 - g(X,P^2\mathrm{X}), \forall X \in \Gamma(TN). \quad (7)$$

Proposition 11. If N is a generalized skew semi-invariant submanifold of an almost paracontact manifold (M,F,ξ,η,g), then $\Delta^0 = KerP$ and $\Delta^1 = KerQ \cap D.$

Proof. The first equality results from the symmetry of P.

If $X \in \Gamma(T_p N)$ is a tangent vector to N at a $p \in N$, then we have to prove the equivalence:

$$(i) X \in \Delta^1 \Leftrightarrow (ii) X \in KerQ \text{ and } (iii) X \in D.$$

If we assume (i), which means:

$$P^2 X = X,$$

then (7) becomes:

$$\|QX\|^2 = -(\eta(X))^2,$$

which shows that both QX and $\eta(X)$ vanish, *i.e.* (ii) and (iii).

Conversely, if we assume (ii) and (iii), then (5) becomes:

$$\|X\|^2 - g(X, P^2 \mathrm{X}) = 0 \tag{8}$$

Since P^2 is symmetric, then there exists an orthonormal basis $\{e_i\}_i$ in $T_p N$, of eigenvectors of P^2, corresponding to distinct eigenvalues $\{\nu_i^2(p)\}_i$. If we write:

$$X = \sum_i X_i e_i$$

then (6) becomes:

$$\sum_i X_i^2 (1 - \nu_i^2(p)) = 0.$$

From Definition 9, one has $\nu_i \in [0,1], \ \forall i,$ which yields that P^2 has only eigenvalue $\nu^2 = 1$ Hence, we obtain (i), which completes the proof.

From Proposition 11, there follows:

Lemma 12. Let N be a generalized skew semi-invariant submanifold of an almost paracontact manifold (M, F, ξ, η, g). If the Reeb vector field ξ is tangent to N, then $\Delta^0 \bigcap KerQ = span\{\xi\}$.

Remark 13. When ξ is tangent to N, the above Lemma yields $span\{\xi\} \subseteq \Delta^0$, which gives the following orthogonal decomposition:

$$\Delta^0 = (span\{\xi\})^\perp \oplus span\{\xi\} \tag{9}$$

where $(span\{\xi\})^\perp$ denotes the orthogonal complement of $span\{\xi\}$ in Δ^0. Hence, $(span\{\xi\})^\perp$ is contained in the paracontact distribution, *i.e.* $(span\{\xi\})^\perp \subseteq D$, or equivalently:

$$\eta((span\{\xi\})^\perp) = 0$$

Definition 14. Let (M,g) be a manifold and let ∇ be a Levi-Civita connection on M. The distiribution ∇ on M is called parallel with respect to a vector field $W \in \Gamma(TM)$, if it is parallel with respect to the distribution $span\{W\}$, *i.e.*:

$$\nabla_Q Y \in \Delta, \forall Y \in \Gamma(\Delta)$$

Proposition 15. Let N be a generalized skew semi-invariant submanifold of an almost paracontact manifold (M,F,ξ,η,g) with ξ tangent to N and Δ^0 parallel with respect to ξ Then, any integral curve of ξ is a geodesic on N if and only if $(span\{\xi\})^{\perp}$ is parallel with respect to ξ (where parallelism is considered with respect to the Levi-Civita connection ∇ on N).

Proof. When ξ is tangent to N, we assume that Δ^0 is parallel with respect to ξ, *i.e.* $\nabla_\xi X \in \Delta^0, \forall X \in \Gamma(\Delta^0)$. In particular:

$$\nabla_\xi \xi, \nabla_\xi W \in \Gamma(\Delta^0), \forall W \in \Gamma((span\{\xi\})^{\perp}). \tag{10}$$

Remark 13, Definition 14 and (10) yield, for any $W \in \Gamma((span\{\xi\})^{\perp})$, the following equivalence:

$(span\{\xi\})^{\perp}$ is parallel with respect to $\xi \Leftrightarrow \nabla_\xi W \in (span\{\xi\})^{\perp} \Leftrightarrow g(\nabla_\xi W, \xi) =$
$= 0 \Leftrightarrow g(W, \nabla_\xi \xi) = 0 \Leftrightarrow \nabla_\xi \xi \in span\{\xi\}$.

As ξ is unitary, one has obviously:

$$g(\nabla_\xi \xi, \xi) = 0. \tag{11}$$

From (10) and (11), there follows that $(span\{\xi\})^{\perp}$ is parallel with respect to ξ if and only if $\nabla_\xi \xi = 0$, which completes the proof.

Example 16. Let M be a 5-dimensional Lie group, with the Riemannian metric g and let $B = \{X_1, ..., X_5\}$ be a global frame of the orthonormal vector fields on M. Let N be generated by $\{X_1, X_2, X_3\}$. We take $\xi = X_1$ and η its dual form. With respect to B we define:

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

hence, $(F,\xi,\eta.g)$ is an almost paracontact structure on M. Since:

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$P^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to X_1, X_2, X_3, there follows that:

$$TN = \Delta^0 \oplus \Delta^1$$

with $\Delta^0 = span\{\xi\}, \Delta^1 = span\{X_2, X_3\}$, all these vector fields being restricted to N. Therefore, N is a generalized skew semi-invariant submanifold of M.

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