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MATHEMATICS

FIBER-PRESERVING CONFORMAL VECTOR FIELD OF FRAME BUNDLES WITH NATURAL RIEMANNIAN METRIC

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We consider the bundle of all oriented orthonormal frames over an orientable Remannian manifold. This bundle has a natural Riemannian metric which is defined by the Riemannian connection of the base manifold. The purpose of the present paper is to classify the conformal vector field structure of the frame bundle with natural Riemannian metric.

Keywords: bundle of all oriented orthonormal frames, conformal vector field

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1. CONFORMAL VECTOR FIELD

A smooth vector field ξ on a Riemannian manifold (M, g) is said to be a conformal vector field if there exists a smooth function f on M that satisfies

$$L\xi\, g = 2f\, g \qquad (1)$$

where $\mathsf{L}_\xi\, g$ s the Lie derivative of g with respect to ξ, that is the flow of the vector field ξ (consisting of conformal transformations of the Riemannian manifold (M, g),) function f being called the potential function of the conformal vector field ξ. We consider ξ a non trivial conformal vector field if ξ is a non-Killing conformal vector field. If the conformal vector field ξ is a closed vector field, then ξ is said to be a closed conformal vector field.

2. INTRODUCTION

Let $(M, <,>)$ be a connected orientable Riemannian manifold of dimension $n \geq 3$ and $SO(M)$ be the bundle of all oriented orthonormal frames over M. $SO(M)$ has a Riemannian metric, also denoted by $<,>$, defined naturally as follows: At each point u of $SO(M)$, the tangent space $(SO(M))_u$ is a direct sum $Q_u + V_u$, where Q_u is the horizontal space defined by the Riemannian connection and V_u is the space of vectors tangent to the fibre through u. The right action of the special orthogonal group $SO(n)$ on the bundle $SO(M)$ gives an isomorphism f_u of the Lie algebra $\mathbf{o}(n)$ onto V_u for each $u \in SO(M)$. We denote by A_u the image of $A \in \mathbf{o}(n)$. On the other hand, $SO(n)$ has a bi-invariant metric denoted also by $<,>$, which is defined by:

$$< A,C >= \sum_{i,j=1}^{n} A_{ij} C_{ij}$$

Then, the Riemannian metric $<,>$ of $SO(M)$ is defined by:

$$\begin{aligned} &< A_u, C_u >=< A, C >, \\ &< A_u, X_u > 0 \\ &< X_u, Y_u >=< X, Y >, \end{aligned} \tag{2}$$

for every $u \in SO(M)$, A, $C \in \mathbf{o}(n)$, X_u, Y_u, Q_u,

Hitoshi Takagi and Makoto Yawata ([1]) studied the Killing vector field of $(SO(M), <,>)$. In the present paper, we shall study Conformal vector fields on $(SO(M), <,>)$ and prove the following Theorems A and B.

Let X be a vector field on $SO(M)$. X is said to be vertical (resp. horizontal) if X_u V_u (resp. if $X_u \in Q_u$) for all $u \in SO(M)$. X is said to be fibre preserving if $[X, Y]$ is vertical for any vertical vector field Y. Let A^* be the vertical vector field defined by $(A^*)_u = A_u = f_u(A)$. A^* is called the fundamental vector field corresponding to $A \in \mathbf{o}(n)$. X is decomposed uniquely as $X = X^H + X^V$, with X^H horizontal and X^v vertical. X^H and X^v are called the horizontal and the vertical part of X, respectively. Let φ be a 2 form on M. Then the tensor field F of type (1, 1) is defined by $<F(Y), Z>= \varphi(Y, Z)$. Then, for each $u \in SO(M)$, $F^q(u) \in \mathbf{o}(n)$ is defined by:

$$F^q(u) = u^{-1} \circ F_{p(u)} \circ u,$$

where X is defined by $X = f_u(F^q(u))$, $u \in SO(M)$. X is called the natural lift of φ or F and is denoted by φ^L or F^L. We denote by $C(M)$ and $C(SO(M))$ the Lie algebras of all Conformal vector fields on M and $SO(M)$, respectively.

For a vector field Y on M, we define a vector field Y^H on $SO(M)$ by $p(Y^H) = Y \cdot Y^H$ is called the horizontal lift of Y. Let Y be a Killing vector field on M and DY the covariant differential of Y. We denote the vector field $Y^H + (DY)^L$ by $Y^L \cdot Y^L$ is called the natural lift of Y.

3. PRELIMINARIES

In this section, we give definitions, notation and lemmas needed to prove Theorems 1.

For $\xi \in \mathbb{R}^n$, we define the standard horizontal vector field $B(\xi)$ on $SO(M)$ by $p(B_u(\xi)) = u(\xi)$, $u \in SO(M)$. We denote also by D the covariant differentiation with respect to the Riemannian connection of $(SO(M), <,>)$.

The proof of the following lemma can be found in ([2]) and ([3]).

Lemma 1. *Let $A, C \in \mathbf{o}(n)$, $\xi, \eta, \zeta \in \mathbb{R}^n$ and let Ω be the curvature form of the Riemannian connection of $(M, <,>)$. Then,*

$$[A^*, C^*] = [A, C]^*, \quad [B(\xi), A^*] = B(A\xi), \quad <[B(\xi), B(\eta)], B(\zeta)> = 0$$

$$<[B(\xi), B(\eta)], A^*> = -2 <\Omega(B(\xi), B(\eta)), \mathrm{A}>, \quad <D_{B(\xi)}B(\eta), B(\zeta)> = 0$$

$$<D_{B(\xi)} B(\eta), A^*> = -<\Omega\,(B(\xi), B(\eta)), A>$$

$$<D_{B(\xi)}A^*, B(\eta)> = <\Omega(B(\xi), B(\eta)), A>$$

$$<D_A * B(\xi), B(\eta)> = <\Omega(B(\xi), B(\eta)), A> + <B(A\xi), B(\eta)>$$

$$<D_A * B(\xi); C^*> = 0, \quad D_A * C^* = \frac{1}{2}[A, C]^*, \quad <D_{B(\xi)}A^*, C^*>= 0$$

Let X be a vector field on $SO(M)$. Then, X is defined by:

$$x(\xi) =<X, B(\xi)>, \xi \in \mathbb{R}^n$$

$$x(A) =<X, A^*>, A \in \mathbf{o}(n)$$

$x(\xi)$ and $x(A)$ are called the ξ-component and the A-component of X, respectively. X is horizontal if and only if $x(A) = 0$ for all $A \in \mathbf{o}(n)$, while X is vertical if and only if $x(\xi) = 0$ for all $\xi \in \mathbb{R}^n$.

Lemma 2. *Let X be a vector field on $SO(M)$. Then,*

$$<[B(\xi), X], B(\eta)> + <[B(\eta), X], B(\xi)> = B(\xi)(x(\eta)) + B(\eta)(x(\xi))$$

$$<[A^*, X], B(\xi)>= A^*(x(\xi)) - x(A\xi)$$

$$<[B(\xi), X], A^*>= B(\xi)(x(A)) - 2 <\Omega(B(\xi), X^H), A>$$

$$<[A^*, X], C^*>= A^*(x(C)) - x([A, C])$$

$$<[B(\xi), X], B(\eta)>= B(\xi)(x(\eta))- <fu-1((X^V)_u)\xi, \eta>$$

Lemma 3. *Let X be the vector field on $SO(M)$. Then, X is a conformal vector field if and only if*

$$B(\xi)(x(\eta)) + B(\eta)(x(\xi)) = 2h(u) <\xi, \eta>,$$

$$A^*(x(\xi)) + B(\xi)(x(A)) = x(A\xi) + 2 <\Omega(B(\xi), X^H), A>, \qquad (3)$$

$$A^*(x(C)) + C^*(x(A)) = 2h(u) <A, C>,$$

with h as a C^∞ function on $SO(M)$ for all $\xi, \eta \in \mathbb{R}^n$ and $A, C \in \mathbf{o}(n)$.

Proof: X is a Conformal vector field if and only if there exists a smooth function h on $SO(M)$ which satisfies

$$<D_{B(\xi)}X, B(\eta)> + <D_{B(\eta)}X, B(\xi)> = 2h(u) <u(\xi), u(\eta)>,$$
$$<D_A * X, B(\xi)> + <D_{B(\xi)}X, A^*> = 0 \tag{4}$$
$$<D_A * X, C^*> + <D_C * X, A^*> = 2h(u) <A, C>,$$

where we have $x(\xi) = <X, B(\xi)>$ and $D_{B(\xi)}B(\eta) = 0$ then $<D_{B(\xi)}X, B(\eta)> = B(\xi)(x(\eta))$ so we have $B(\xi)(x(\eta) + B(\eta)(x(\xi) = 2h(u) <u(\xi), u(\eta)>$

On the other hand, $x(A) = <X, A^*>$ and $D_{A^*} C^* + D_{C^*} A^* = 0$, then we have

$$A^* (x(C)) + C^* (x(A)) = 2h(u) <A, C>.$$

For the third equation where we apply lemma (3), we have:

$$<DB(\xi)X, A*>=<\Omega(B(\xi), Xh), A>$$

$$<DA* X, B(\xi)>=<B(A\xi), X> + <\Omega(B(\xi), Xh), A>.$$

Hence, we have:

$$A* (x(\xi)) + B(\xi)(x(A)) = x(A\xi) + 2 <R(B(\xi), XH), A>$$

4. FIBER-PRESERVING CONFORMAL VECTOR FIELD

Y^L (resp φ^L) is a Killing vector field if and only if Y is a Killing vector field (resp φ is a parallel 2-form) [2]

We are now looking for a necessary and sufficient condition for Y^L (resp φ^L) to be a Conformal vector field.

Lemma 4. *Let $X = Y^L$. X is a conformal vector field if and only if X is a Killing vector field if and only if Y is a Killing vector field.*

Proof: Let g be an R^n-valued function on $SO(M)$ defined by $g(u) = u^{-1}Y_{p(u)}$. Then, we have $D_{u(\xi)}Y = u(B(\xi)g)$ for all ξ $\mathrm{R}^n \in$ and u $SO(M)$. For the proof, see the lemma of section 1 of chapter *III* of [2]. It follows that:

$$x(\xi)(u) =<X, Bu(\xi)>=<p(XH), p(Bu(\xi))>=<Yp(u), u(\xi)>$$

Moreover, $B_u(\eta)(x(\xi)) =<B_u(\eta)g, \xi> = <D_{u(\eta)}Y, u(\xi)>$.

Then, by Lemma 3, for all $\xi, \eta \in \mathrm{R}^n$ and $u \in SO(M)$,

$$<Du(\eta)Y, u(\xi)> + <Du(\xi)Y, u(\eta)>= 2h(u) <u(\xi), u(\eta)> \tag{5}$$

we put $X_1 = (DY)^L$ and $F = DY$ and let F^* be the $\mathbf{o}(n)$-valued function on $SO(M)$ defined by $F^q = u \circ F_{p(u)} \circ u$. Then $F^* (ua) = a^{-1} \circ F^q (u) \circ a$ for all $a \in SO(M)$, which implies that $R_a(X_1)u = (X_1)_{ua}$ for all $a \in SO(n)$, that is, $[A^*, X_1] = 0$ for all $A \in \mathbf{o}(n)$. Then $A^* (x_1(C)) = x_1([A, C])$ then:

$$0 = A* (x1(C)) + C* (x1(A)) = 2h(u) <A, C>$$

then $h = 0$, so Y^L is a Killing vector field

Now, let X be a fibre preserving Conformal vector field.

Theorem 1. *Let X be a fibre preserving Conformal vector field on* $(SO(M), <,>)$*. Then, X is a fibre preserving the Killing vector field on* $(S0(M), <,>)$*, being decomposed as:*

$$X = Y L + \varphi L + A^*$$

where Y^L *is the natural lift of a Killing vector field Y on* $(M, <,>)$, φ^L *is the natural lift of a parallel* 2*-form* φ *on* $(M, <,>)$ *and* A^* *is the fundamental vector field.*

Lemma 5. *Let X be a fibre preserving Conformal vector field on* $(SO(M), <,>)$*. Then, there exists a Killing vector field Y on* $(M, <,>)$ *so that* $X^H = (Y^L)^H$ *and* Y^L *is a Killing vector field.*

Proof: Since X^v is fibre preserving, so is X^H and, by Lemma 2:

$$<[A^*, XH], B(\xi)>=<[A^*, XH], C^*> = 0 \tag{6}$$

for all A, $C \in \mathbf{o}(n)$ and $\xi \in \mathrm{R}^n$. It means that $[A^*, X^H] = 0$ for all $A \in \mathbf{o}(n)$ that is, $R_a X^H = X^H$ for all a $S \in O(n)$. Hence, there exists a unique vector field Y on M satisfying $p(X^H) = Y$.

By lemma, Y^L is a Killing vector field. Then, $X - Y^L$ is a vertical Conformal vector field and $X - Y^L$ is a Killing vector field, then X is a Killing vector field.

Theorem 2. *X is a horizontal conformal vector field if and only if X is a Killing vector field, X is a vertical conformal vector field if and only if X is a Killing vector field.*

Proof: If we suppose that X is a conformal vector field on $(SO(M), <,>)$, then there is a C^∞-function h on $SO(M)$, so that:

$$\begin{gathered} B(\xi)(x(\eta)) + B(\eta)(x(\xi)) = 2h(u) <\xi, \eta>, \\ A^*(x(\xi)) + B(\xi)(x(A)) = x(A\xi) + 2 <\Omega(B(\xi), XH), A>, \\ A^*(x(C)) + C^*(x(A)) = 2h(u) <A, C>. \end{gathered} \tag{7}$$

If X is a horizontal vector field, then $x(A) = 0$ for all $A \in \mathbf{o}(n)$. In particular from the third equation of (4.3), we obtain $h(u) <A, C>= 0$ for all A, $C \in \mathbf{o}(n)$. Taking $C = A$ with $A = 1$, we deduce that h vanishes, i.e. X is a Killing vector field on $(SO(M), <,>)$.

If X is a vertical vector field, then $x(\xi) = 0$ for all ξ R^n. In particular from the first equation of (4.3), we obtain $h(u) <\xi, \eta> = 0$ for all η, ξ, R^n. Taking $\xi = \eta$ with $\xi = 1$, we deduce that h vanishes, i.e. X is a Killing vector field on $(SO(M), <,>)$.

Corollary 1. *[1] If* $(SO(M), <,>)$ *has a horizontal conformal vector field which is not fibre preserving, then it has a constant curvature* $\frac{1}{2}$, *except when* $\dim M = 3, 4$ *or* 8.

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