Memoirs of the Scientific Sections of the Romanian Academy
Tome XLII, 2019

MATHEMATICS

ON SUBMANIFOLDS OF RIEMANNIAN MANIFOLDS ADMITTING A RICCI SOLITON

SEMSI EKEN MERIC[1] **and EROL KILIC**[2]

[1] *Mersin University, Department of Mathematics, Mersin, TURKEY*
[2] *İnönü University, Department of Mathematics, Malatya, TURKEY*
Corresponding author: semsieken@hotmail.com, semsieken@mersin.edu.tr

The aim of this paper is to study the conditions under which a submanifold of a Ricci soliton is also a Ricci soliton or an almost Ricci soliton. We give here a classification for Ricci solitons and their submanifolds according to their expanding, shrinking or steady cases.

Kew words: Ricci soliton, Riemannian submanifold, curvature

1. INTRODUCTION

Let (M, g) be a Riemannian manifold. A triple (M, g, V) is called a Ricci soliton, if it satisfies:

$$\frac{1}{2}L_V g + Ric + \lambda g = 0 \tag{1}$$

V is a vector field on M, Ric is the Ricci tensor of (M, g) and V is a potential vector field of the Ricci soliton. Here, $L_V g$ is the Lie derivative of the metric tensor g with respect to V and λ is a constant.

A Ricci soliton (M, g, V) is said to be shrinking, steady or expanding if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. Moreover, if the potential vector field V is zero, Killing or concurrent, then the Ricci soliton is an Einstein. Hence, Ricci solitons can be viewed as generalizations of Einstein manifolds.

A Ricci soliton (M, g, V) is a gradient Ricci soliton, if there exists a smooth function f on M such that $V = \nabla f$. In this case, the triple (M, g, f) is a gradient Ricci soliton and we refer to f as the potential function of the soliton. Obviously, a gradient Ricci soliton (M, g, f) is trivial, if its potential function f is a constant.

The notion of Ricci flow was first introduced by Hamilton in 1982. According to the definition of Hamilton, a Ricci flow can be given as:

$$\frac{\partial}{\partial t} g_{ij} = -2Ric_{ij}$$

On the other hand, the concept of Ricci soliton appeared after Hamilton introduced such above flow in 1982, since the Ricci soliton is the geometric fixed point (modulo homothetics and differomorphisms) of the Ricci flow. Also, the Ricci soliton appears as singular model for such a flow, analyzing its geometry is useful step towards an understanding of the Ricci flow itself. (for details, see [10,11]).

In 2011, Barros and *et al.* studied the immersions of a Ricci soliton (M, g, V) into a Riemannian manifold $(\bar{M}, g)$ and showed that a shrinking Ricci soliton immersed into a space form with constant mean curvature must be a Gaussian soliton [1]. In [8], Chen and Deshmukh classified the Ricci solitons whose potential vector fields are concurrent, and provided a necessary and sufficient condition for such a submanifold to be a Ricci soliton in a Riemannian manifold.

On the other hand, many papers on a Riemannian manifold endowed with some geometric structures so that this manifold admits a Ricci soliton were discussed by several mathematicians (for details, we refer to [2–5,7,9,13]).

Motivated by the above studies, our paper is organized as follows: Section 2, contains some preliminaries. The next section is devoted to the submanifold of Riemannian manifold admitting a Ricci soliton. We point out the conditions under which a submanifold of a Ricci soliton is also a Ricci soliton or almost Ricci soliton. In Section 4, a relationship between the intrinsic and extrinsic invariants (such as scalar curvature and squared mean curvature) of a Riemannian submanifold admitting a Ricci soliton is given. Moreover, we establish some inequalities to obtain characterizations involving such invariants about a submanifold of a Ricci soliton.

2. PRELIMINARIES

2.1. Basic Formulas and Definitions for Submanifolds

Let $(\bar{M}, \bar{g})$ be an $m-$dimensional Riemannian manifold and $\varphi : M \to \bar{M}$ be an isometric immersion from an $n-$dimensional Riemannian manifold (M, g) into $(\bar{M}, \bar{g})$.

The Levi-Civita connections of ambient manifold $\bar{M}$ and the submanifold M will be denoted by $\bar{\nabla}$ and ∇, respectively.

To fix notations, we write here the Gauss-Weingarten formulas:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2}$$

$$\overline{\nabla}_X V = -A_X V + \nabla^{\perp}_X V \tag{3}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$.

For any $V \in \Gamma(TM^{\perp})$,the shape operator and the second fundamental form are related by:

$$\overline{g}(h(X,Y),V) = g(A_X V, Y).$$

The mean curvature vector field H of M in $\overline{M}$ is given by:

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$$

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis of submanifold M.

Moreover, submanifold M is totally umbilical if and only if:

$$h(X, y) = g(X,Y)H \tag{4}$$

for any $X, Y \in \Gamma(TM)$.

The equations of Gauss and Codazzi are given by the following relation

$$g(R(X,Y)Z,W) = \overline{g}(\overline{R}(X,Y)Z,W) + \overline{g}(h(X,W),h(Y,Z)) - \overline{g}(h(X,Z),h(Y,W)) \tag{5}$$

$$(\overline{R}(X,Y)Z)^{\perp} = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z),$$

for any $X, Y, Z, W \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$. Here, $(\overline{R}(X,Y)Z)^{\perp}$ is the normal component of $\overline{R}(X,Y)Z$ and $\overline{\nabla}h$ is given by:

$$(\overline{\nabla}_X, h)(Y,Z) = (\nabla^{\perp}_X h)(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y). \tag{6}$$

Let P be a 2-plane section spanned by orthonormal vectors X and Y. From (5), one has:

$$K(P) = \overline{K}(P) - \|h(X,Y)\|^2 + g(h(X,X), h(Y,Y)).$$

The Ricci tensor Ric is defined by:

$$Ric(X,Y) = \sum_{j=1}^{n} R(e_j, X, Y, e_j) \tag{7}$$

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis of M and R is the Riemannian curvature tensor of M, for any $X, Y \in T_pM$.

Furthermore, the Ricci tensor Ric on $\overline{M}$ can be written as:

$$\overline{Ric}(X,Y) = \overline{Ric}\Big|_{T_pM}(X,Y) + \overline{Ric}\Big|_{T_pM^{\perp}}(X,Y)$$

for any $X, Y \in T_pM$. Throughout this paper, we assume that the normal part of Ricci tensor $\overline{Ric}$ vanishes identically.

On the other hand, the divergence of any $X \in \Gamma(TM)$ is denoted by $div(X)$, being given as:

$$div(X) = \sum_{i=1}^{n} g(\nabla_{e_i} X, e_i). \tag{8}$$

For details, we refer to [6].

3. SOME PROPERTIES OF THE SUBMANIFOLD OF A RIEMANNIAN MANIFOLD ADMITTING A RICCI SOLITON

The Ricci solitons are natural extensions of Einstein manifolds and self-similar solutions to their Ricci flow equation, as proved by Hamilton in [10]. In [12], Pigola *et al*. introduced a new class of Ricci solitons, called almost Ricci soliton, by taking in equation (1) λ as a function. If function λ is a constant, then the almost Ricci soliton becomes Ricci soliton. Similarly, we say that the almost Ricci soliton is shrinking, steady or expanding if the variable function λ is negative, zero or positive, respectively.

Assumption: Let $(\overline{M}, \overline{g}, V)$ be a Ricci soliton and $\varphi : M \to \overline{M}$ be an isometric immersion. Throughout the present paper, we take the potential vector field V tangent to the submanifold M.

We may characterize above notions as follows:

Theorem 1. Let $(\overline{M}, \overline{g}, V)$ be a Ricci soliton and M be a totally umbilical submanifold of $\overline{M}$. In this case, the following conditions are satisfied:

i) M is an almost Ricci soliton.

ii) If M has a constant mean curvature, then M becomes a Ricci soliton.

Proof: Since $\overline{M}$ is a Ricci soliton, one has:

$$\begin{aligned}\frac{1}{2}L_V\bar{g}(X,Y)+\overline{Ric}(X,Y)+\lambda\bar{g}(X,Y) &= \frac{1}{2}\left\{\bar{g}(\bar{\nabla}_X V,Y)+\bar{g}(\bar{\nabla}_Y V,X)\right\}\\ &\quad+\overline{Ric}(X,Y)+\lambda\bar{g}(X,Y)\\ &= \frac{1}{2}\left\{g(\nabla_X V,Y)+g(\nabla_Y V,X)\right\}\\ &\quad+Ric(X,Y)+\lambda g(X,Y)\\ &= 0,\end{aligned}$$

for any $X,Y\in\Gamma(TM)$. Using equalities (5) and (7), there follows:

$$\overline{Ric}(X,Y)=Ric(X,Y)-\sum_{i=1}^{n}\left\{\bar{g}(h(e_i,e_i),h(X,Y))+\bar{g}(h(X,e_i),h(Y,e_i))\right\}, \tag{9}$$

where $\{e_1,e_2,...,e_n\}$ denotes an orthonormal basis of T_pM, at $p\in M$. Since M is totally umbilical submanifold, considering (4) one has:

$$\bar{g}(h(X,e_i),h(Y,e_i))=\bar{g}(X,Y)\|H\|^2. \tag{10}$$

According to (9) and (10), there follows that:

$$\begin{aligned}&\overline{Ric}(X,Y)=Ric(X,Y)-\bar{g}(nH,g(X,Y)H)+\bar{g}(g(X,e_i)H,g(Y,e_i)H)=\\ &=Ric(X,Y)-n\|H\|^2g(X,Y)+\|H\|^2g(X,Y)=Ric(X,Y)+(1-n)^2\|H\|^2g(X,Y)\end{aligned} \tag{11}$$

From (1) and (11), we have:

$$\frac{1}{2}L_V g(X,Y)+Ric(X,Y)+\left\{(1-n)^2\|H\|^2+\lambda\right\}g(X,Y)=0 \tag{12}$$

which means that submanifold M is an almost Ricci soliton. Also, assume that M has a constant mean curvature. From equation (12), submanifold M becomes a Ricci soliton, which proves (i) and (ii).

Recall that the position vector field V of an Euclidean space is a concurrent vector field. Here, suppose that an Euclidean space E^6 is endowed with a concurrent vector field V. For an isometric immersion $\varphi:M\to E^6$, we denote by V^T and $V^\perp$ the tangent and normal parts of V on $\bar{M}$, respectively. Inspired by some examples of [8], we construct some examples for submanifold of a Ricci soliton which provide Theorem 1, as follows:

Example 2. Let $\bar{M}=S^2(1)\times E^3$ be a hypersurface of an Euclidean space E^6 with coordinates $\left(x_1,x_2,\sqrt{1-(x_1^2+x_2^2)},x_4,x_5,x_6\right)$. Then, $\left(\bar{M},\bar{g},V^T\right)$ is a

Ricci soliton with potential vector field V^T, so that V^T is the tangential part of the position vector of E^6 and $\bar{g}$ is a product metric. For details, we refer [8] (see Theorem 4.1, pp.17).

Now, we consider a submanifold M of Ricci soliton $\left(\bar{M}, \bar{g}, V^T\right)$ with codimension 1 (*i.e.* M is a hypersurface of $\left(\bar{M}, \bar{g}, V^T\right)$ with the position vector:

$$\left(x_1, x_2, \sqrt{1-(x_1^2+x_2^2)}, x_4, x_5, x_6\right)$$

so that:

$$x_1 = x_2.$$

Then, the tangent bundle TM and normal bundle $TM^{\perp}$ are spanned by:

$$E_1 = x_3 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3}, \qquad E_2 = \frac{\partial}{\partial x_4},$$

$$E_3 = \frac{\partial}{\partial x_5}, \qquad E_4 = \frac{\partial}{\partial x_6}$$

and:

$$N = x_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2},$$

respectively. By direct calculations, one can see that submanifold M is totally geodesic and the mean curvature vector field H vanishes identically. Therefore, it is obvious that Theorem 1 is satisfied.

Example 3. Let $\gamma(s)$ be a unit speed curve lying on the hypersphere $S_0^2(1)$ of E^3 centered of the origin o. Consider the hypersurfaces of $(\bar{M}, \bar{g})$ of $S_0^2(1) \times E^6$ is defined by:

$$\phi(s, x_2, x_3, x_4, x_5, x_6) = (\gamma(s), x_2, x_3, x_4, x_5, x_6).$$

Then, $(\bar{M}, \bar{g}, V^T)$ is a Ricci soliton with potential vector is V^T which is the tangential part of position vector V of E^8 and $\lambda = 1$. For details, we refer [8] (see Theorem 4.1, pp.17).

Now, we consider that a submanifold M of Ricci soliton $(\bar{M}, \bar{g}, V^T)$ is defined by:

$$x_2 = 0, \quad x_3 = t, \quad x_4 = \cos u \cos v,$$

$$x_5 = \sin u \cos v, \quad x_6 = \sin v.$$

Then, the tangent bundle TM is spanned by $\{E_1, E_2, E_3\}$, as follows:

$$E_1 = \frac{\partial}{\partial x_3},$$

$$E_2 = -\sin u \frac{\partial}{\partial x_4} + \cos u \frac{\partial}{\partial x_5},$$

$$E_3 = -\cos u \sin v \frac{\partial}{\partial x_4} - \sin u \sin v \frac{\partial}{\partial x_5} + \cos v \frac{\partial}{\partial x_6}.$$

Similarly, the normal bundle $TM^{\perp}$ is spanned by $\{N_1, N_2, N_3\}$, as follows:

$$N_1 = \gamma'(s) \frac{\partial}{\partial x_1},$$

$$N_2 = \gamma(s) \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},$$

$$N_3 = \cos u \cos v \frac{\partial}{\partial x_4} + \sin u \cos v \frac{\partial}{\partial x_5} + \sin v \frac{\partial}{\partial x_6}.$$

By direct calculations, we obtain:

$h(E_1, E_1) = 0$, $h(E_1, E_2) = 0$, $h(E_2, E_2) = -\cos v N_3$, $h(E_1, E_3) = 0$, and $h(E_3, E_3) = -N_3$, then the mean curvature vector field $H = -N_3$, which means that M is totally umbilical – which permits to directly verify that Theorem 1 is satisfied.

Using equality (12), we get the following:

Corollary 4. Let $(\bar{M}, \bar{g}, V)$ be a Ricci soliton and M be a totally umbilical submanifold of $\bar{M}$. Then, the following conditions are satisfied:

i) If M has a constant mean curvature vector and the Ricci soliton $(\bar{M},\bar{g},V)$ is shrinking or steady (i.e. $\lambda \le 0$), then the almost Ricci soliton (M,g,V) is shrinking.

ii) Suppose that M is minimal (namely, the mean curvature vector H) vanishes identically). Then, one has the following situations:

a) If $(\bar{M},\bar{g},V)$ is shrinking, then (M,g,V) is shrinking.

b) If $(\bar{M},\bar{g},V)$ is steady, then (M,g,V) is steady.

Remark 5. Let $(\bar{M},\bar{g},V)$ be a Ricci soliton and M a totally umbilical submanifold of $\bar{M}$, so that M has a constant mean curvature. If the potential vector field V is concurrent on $\bar{M}$ (namely, $\bar{\nabla}_X V = X$ for any $X \in \Gamma(T\bar{M})$), then M is Einstein.

Acknowledgements. This work is supported by the 1001-Scientific and Technological Research Projects Funding Program of TUBITAK project number 117F434.

R E F E R E N C E S

1. BARROS, A., VIEIRA GOMES, J.N., RIBEIRO, E. *Immersion of almost Ricci solitons into a Riemannian manifold*, Math. Cont., 2011, 40, 91–102.
2. BEJAN, C.L., CRASMAREANU, M., *Ricci solitons in manifolds with quasi-constant curvature*, Publ. Math. Debrecen, 2011, 78 (1), 235–243.
3. BEJAN, C.L., CRASMAREANU, *Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry*, Anal. Glob. Anal. Geom., 2014, 46 (2), 117–128.
4. BLAGA, A.M., PERKTAŞ S.Y. , *Remarks on almost η – Ricci solitons in (ε)-para Sasakian manifolds*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 2019, 68 (2), 1621–1628.
5. BLAGA, A.M., PERKTAŞ S.Y., ACET, B.E., ERDOĞAN, F.E., *η – Ricci solitons in (ε)-almost paracontact metric manifolds*, Glasnik Math., 2018, 1, 377–410.
6. BESSE, A.L. *Einstein manifolds*, Berlin-Heidelberg-NewYork: Spinger-Verlag, p.516, 1987.
7. CHEN, B.Y., *Concircular vector fields and pseudo-Kahler manifolds*, Kragujevac J. Math., 2016, 40(1), 7–14.
8. CHEN, B.Y., DESHMUKH, S., *Ricci solitons and concurrent vector Fi*eld, Balkan J. Geom. Its Appl., 2015, 20 (1), 14–25.
9. EKEN MERİÇ, Ş., KILIÇ, E., *Some Inequalities for Ricci Solitons*, Proceedings of International Conference on Mathematics and Mathematics Education, Turk. J. Math. Comp. Sci., 2018, 10, 160–164.
10. HAMILTON, R.S., *The Ricci flow on surfaces, Mathematics and General Relativity* (Santa Cruz, CA, 1986) Contemp. Math. Amer. Math. Soc., 1988, 71, 237–262.
11. PERELMAN, G., *The Entropy formula for the Ricci flow and its geometric applications*, 2002, arXiv math/0211159.
12. PIGOLA, S., RIGOLI, M., RIMOLDI, M., SETTI, A.G., *Ricci almost solitons*, Ann. Scuola Norm. Sup. Pisa. Cl. Sci. 2011, 10 (4), 757–799.
13. PERKTAŞ, S.Y., KELEŞ, S., *Ricci solitons in 3-dimensional normal almost paracontact metric manifolds*, Int. Elect. J. Geom. 2015, 8 (2), 34–45.