Memoirs of the Scientific Sections of the Romanian Academy
Tome XLII, 2019

MATHEMATICS

IDENTITIES
COMPLETE INDUCTION METHOD

GABRIELA CONSTANTIN
Şcoala Gimnazială no. 37, Constanţa

The aim of this project is to present the complete induction method, Gauss Identity, the method of non-determinant coefficients and recurrence relationships and Newton's binomial.

Keywords: inductor method, Gaussian identity.

1.Many identities are demonstrated using the **complete induction method**. Here are examples of such identities:

i. *To show that:*

$$\mathcal{A}_n = (1+x)(1+x^2)(1+x^{2^2})...(1+x^{2^n}) = 1+x+x^2+...+x^{2^{n+1}-1} \quad (1)$$

where n is a positive integer or zero.

Identity (1) is true for $n = 0$, because we have:

$$\mathcal{A}_0 = 1+x$$

According to the complete induction method, we will assume that identity (1) is true for index n and we will prove that it is still true for index $n+1$.

For this, we can write:

$$\mathcal{A}_{n+1} = \mathcal{A}_n(1+x^{2^{n+1}})$$

and, since it was assumed that identity (1) is true for index n, we can replace it by $\mathcal{A}_n$ with the second member of formula (1). We will have:

$$\mathcal{A}_{n+1} = (1+x+x^2+...+x^{2^{n+1}-1})(1+x^{2^{n+1}})$$

To obtain the product from the second member, we multiply the first parenthesis by 1 and then by $x^{2^{n+1}}$. We will have:

$$\mathcal{A}_{n+1} = 1 + x + x^2 + \ldots + x^{2^{n+1}-1} + x^{2^{n+1}} + \ldots + x^{2^{n+1}-1+2^{n+1}}.$$

We notice that a term in the second member is deduced from the precedent by multiplying it by x, and the exponent of the last term is $2^{n+2}-1$. So we can write:

$$\mathcal{A}_{n+1} = 1 + x + x^2 + \ldots + x^{2^{n+2}-1}$$

$\mathcal{A}_{n+1}$ is deduced by itself from the second member of identity (1) by changing it *n* in *n* + 1, it turns out that identity (1) is valid whatever the whole *n*, positive or null [1].

ii. *Identities useful in solving mutual equations. It is known that the reciprocal equations are solved by making the substitution:*

$$x + \frac{1}{x} = y \tag{2}$$

and by calculating $x^2 + \frac{1}{x^2}, x^3 + \frac{1}{x^3}, \ldots$ with the help of *y*.

Making the square of $x + \frac{1}{x}$; from formula (2), there follows that:

$$x^2 + \frac{1}{x^2} = y^2 - 2. \tag{3}$$

Multiplying both members of this formula by $x + \frac{1}{x}$, we have:

$$x^3 + \frac{1}{x^3} + x + \frac{1}{x} = y^3 - 2y$$

so that:

$$x^3 + \frac{1}{x^3} = y^3 - 3y. \tag{4}$$

Let’s prove that we generally have:

$$\begin{aligned} x^n + \frac{1}{x^n} &= y^n - \frac{n}{1!}y^{n-2} + \frac{n(n-3)}{2!}y^{n-4} - \frac{n(n-4)(n-5)}{3!}y^{n-6} + \ldots + \\ &+ (-1)^p \frac{n(n-p-1)(n-p-2)\ldots(n-2p+1)}{p!}y^{n-2p} + \ldots \end{aligned} \tag{5}$$

the formula ending with $(-1)^q \cdot 2$ *if* $n = 2q$ *and with* $(-1)^q (2q+1)y$ *if* $n = 2q+1$.

Formula (5) is true for $n=2$ and $n=3$, being reduced in these cases to formulas (3) and (4).

According to the complete induction method, let us show that, if formula (5) is assumed to be true for the $n-1$ indices and n, it can be proved that it is still true for the $n+1$ indices.

Let's use the identity:

$$\left(x^n+\frac{1}{x^n}\right)\left(x+\frac{1}{x}\right)=\left(x^{n+1}+\frac{1}{x^{n+1}}\right)+\left(x^{n-1}+\frac{1}{x^{n-1}}\right) \quad (n\geq 1)$$

in which we replace $x+\frac{1}{x}$ with *y*, $x^n+\frac{1}{x^n}$ with the second member of the formula (5), and $x^{n-1}+\frac{1}{x^{n-1}}$ with

$$x^{n-1}+\frac{1}{x^{n-1}}=y^{n-1}-\frac{n-1}{1!}y^{n-3}+\frac{(n-1)(n-4)}{2!}y^{n-5}-...+ \\ +(-1)^{p-1}\frac{(n-1)(n-p-1)(n-p-2)...(n-2p+2)}{(p-1)!}y^{n+1-2p}+... \tag{6}$$

Then, from identity (6) there follows that:

$$x^{n+1}+\frac{1}{x^{n+1}}=y^{n+1}-\frac{n}{1!}y^{n-1}+\frac{n(n-3)}{2!}y^{n-3}-\frac{n(n-4)(n-5)}{3!}y^{n-5}+...-y^{n-1}+ \\ +\frac{n-1}{1!}y^{n-3}-\frac{(n-1)(n-4)}{2!}y^{n-5}+...+ \\ +(-1)^p\frac{n(n-p-1)(n-p-2)...(n-2p+1)}{p!}y^{n+1-2p}+ \\ +...+(-1)^p\frac{(n-1)(n-p-1)(n-p-2)...(n-2p+2)}{(p-1)!}y^{n+1-2p}+... \tag{7}$$

Notice that:

$$\frac{n}{1!}+1=\frac{n+1}{1!}$$

$$\frac{n(n-3)}{2!}+\frac{n-1}{1!}=\frac{(n+1)(n-2)}{2!}$$

$$\frac{n(n-4)(n-5)}{3!}+\frac{(n-1)(n-4)}{2!}=\frac{(n+1)(n-3)(n-4)}{3!}$$

....

and generally:

$$\frac{n(n-p-1)(n-p-2)-(n-2p+1)}{p!}+\frac{(n-1)(n-p-1)(n-p-2)-(n-2p+2)}{(p-1)!}=$$

$$=\frac{(n-p-1)(n-p-2)-(n-2p+2)}{p!}[n(n-2p+1)+p(n-1)]=$$

$$=\frac{(n-1)(n-p)(n-p-1)-(n-2p+2)}{p!}$$

Formula (7) is therefore written:

$$x^{n+1}+\frac{1}{x^{n+1}}=y^{n+1}-\frac{n+1}{1!}y^{n-1}+\frac{(n+1)(n-2)}{2!}y^{n-3}-\frac{(n+1)(n-3)(n-4)}{3!}y^{n-5}+\ldots+$$

$$+(-1)^{p}\frac{(n+1)(n-p)(n-p-1)-(n-2p+2)}{p!}y^{n+1-2p}+\ldots$$

and it coincides with formula (5), in which n is changed into $n+1$.

Formula (5) is thus true, whatever the natural number n [2].

2. GAUSSIAN IDENTITY

Identify the identity:

$$\mathcal{A}_{n}=1-\frac{x^{2n}-1}{x-1}+\frac{\left(x^{2n}-1\right)\left(x^{2n-1}-1\right)}{(x-1)\left(x^{2}-1\right)}-\frac{\left(x^{2n}-1\right)\left(x^{2n-1}-1\right)\left(x^{2n-2}-1\right)}{(x-1)\left(x^{2}-1\right)\left(x^{3}-1\right)}+\ldots-$$

$$-\frac{\left(x^{2n}-1\right)\left(x^{2n-1}-1\right)\ldots\left(x^{2}-1\right)}{(x-1)\left(x^{2}-1\right)\ldots\left(x^{2n-1}-1\right)}+\frac{\left(x^{2n}-1\right)\left(x^{2n-1}-1\right)\ldots(x-1)}{(x-1)\left(x^{2}-1\right)\ldots\left(x^{2n}-1\right)}= \tag{8}$$

$$=(1-x)\left(1-x^{3}\right)\left(1-x^{5}\right)\ldots\left(1-x^{2n-1}\right)$$

whatever is the natural number n, known as Gaussian identity.

For $n=1$, the identity is true, because:

$$\mathcal{A}_{1}=1-\frac{x^{2}-1}{x-1}+\frac{\left(x^{2}-1\right)(x-1)}{(x-1)\left(x^{2}-1\right)}$$

and we have:

$$\mathcal{A}_{1}=1-(x+1)+1=1-x$$

According to the complete induction method, we will assume that identity (8) is true for the index n and we will prove it for the index $n+1$.

Let us note for the abbreviation:

$$\mathcal{B}_n^k = \frac{\left(x^{2n}-1\right)\left(x^{2n-1}-1\right)...\left(x^{2n-k}-1\right)}{\left(x-1\right)\left(x^2-1\right)...\left(x^{k+1}-1\right)} \tag{9}$$

With this notation, we can write:

$$\mathcal{A}_n = 1-\mathcal{B}_n^0+\mathcal{B}_n^1-\mathcal{B}_n^2+...-\mathcal{B}_n^{2n-2}+\mathcal{B}_n^{2n-1} \tag{10}$$

And

$$\mathcal{A}_{n+1} = 1-\mathcal{B}_{n+1}^0+\mathcal{B}_{n+1}^1-\mathcal{B}_{n+1}^2+...-\mathcal{B}_{n+1}^{2n-2}+\mathcal{B}_{n+1}^{2n-1}-\mathcal{B}_{n+1}^{2n}+\mathcal{B}_{n+1}^{2n+1}) \tag{11}$$

By subtracting formulas (10) and (11) from member to member, we have:

$$\begin{aligned}\mathcal{A}_n-\mathcal{A}_{n+1} &= (\mathcal{B}_{n+1}^0-\mathcal{B}_n^0)-(\mathcal{B}_{n+1}^1-\mathcal{B}_n^1)+(\mathcal{B}_{n+1}^2-\mathcal{B}_n^2)-...+\\ &+(-1)^k(\mathcal{B}_{n+1}^k-\mathcal{B}_n^k)+...+(\mathcal{B}_{n+1}^{2n-2}-\mathcal{B}_n^{2n-2})-(\mathcal{B}_{n+1}^{2n-1}-\mathcal{B}_n^{2n-1})+\\ &+(\mathcal{B}_{n+1}^{2n}-\mathcal{B}_n^{2n+1})\end{aligned} \tag{12}$$

We will first prove that:

$$\mathcal{B}_{n+1}^0-\mathcal{B}_n^0 = x^{2n+1}+x^{2n} \tag{13}$$

and then

$$\mathcal{B}_{n+1}^1-\mathcal{B}_n^1 = x^{2n+1}\mathcal{B}_n^0+x^{2n}+x^{2n-1}\mathcal{B}_n^0 \tag{14}$$

and in general

$$\mathcal{B}_{n+1}^k-\mathcal{B}_n^k = x^{2n+1}\mathcal{B}_n^{k-1}+x^{2n-k+1}\mathcal{B}_n^{k-2}+x^{2n-k}\mathcal{B}_n^{k-1} \tag{15}$$

for $k=2,3,...,2n-1$.

Finally, we will show that:

$$\mathcal{B}_{n+1}^{2n} = x^{2n+1}\mathcal{B}_n^{n-1}+x\mathcal{B}_n^{2n-2}+\mathcal{B}_n^{2n-1} \tag{16}$$

and

$$\mathcal{B}_{n+1}^{2n+1} = \mathcal{B}_n^{2n-1} = 1 \tag{17}$$

From formulas (13) to(17), it will be shown that:

$$\mathcal{A}_{n+1} = \left(1-x^{2n+1}\right)\mathcal{A}_n \tag{18}$$

which proves that the formula (8), assumed to be valid for index n, is also valid for index $n+1$.

Formula (13) is obvious because we have:

$$\mathcal{B}_{n+1}^0-\mathcal{B}_n^0 = \frac{x^{2n+2}-1}{x-1}-\frac{x^{2n}-1}{x-1} = x^{2n}\left(x+1\right) = x^{2n+1}+x^{2n}$$

For the formula (14) we have:

$$\mathcal{B}_{n+1}^{1}=\frac{(x^{2n+2}-1)(x^{2n+1}-1)}{(x-1)(x^{2}-1)}$$

$$\mathcal{B}_{n}^{1}=\frac{(x^{2n}-1)(x^{2n-1}-1)}{(x-1)(x^{2}-1)}$$

and hence:

$$\mathcal{B}_{n+1}^{1}-\mathcal{B}_{n}^{1}=\frac{(x^{2n+2}-1)(x^{2n+1}-1)-(x^{2n}-1)(x^{2n-1}-1)}{(x-1)(x^{2}-1)}$$

but:

$$\left(x^{2n+2}-1\right)\left(x^{2n+1}-1\right)-\left(x^{2n}-1\right)\left(x^{2n-1}-1\right)=x^{4n+3}-x^{2n+2}-x^{2n-1}-x^{4n-1}+$$
$$+x^{2n}+x^{2n-1}=x^{4n-1}(x^{4}-1)-x^{2n}\left(x^{2}-1\right)-x^{2n}-1\left(x^{2}-1\right)=$$
$$=\left(x^{2}-1\right)\left[x^{4n+1}+x^{4n-1}-x^{2n}-x^{2n-1}\right]=(x^{2}-1)[x^{2n+1}(x^{2n-1})+x^{2n+1}+$$
$$+x^{2n-1}(x^{2n}-1)+(x^{2n-1}-x^{2n}-x^{2n-1}]=(x^{2}-1)[x^{2n-1}(x^{2n}-1)+x^{2n}(x-1)+$$
$$+x^{2n-1}(x^{2n}-1)]$$

So, we have:

$$\mathcal{B}_{n+1}^{1}-\mathcal{B}_{n}^{1}=x^{2n+1}\frac{x^{2n}-1}{x-1}+x^{2n}+x^{2n-1}\frac{x^{2n}-1}{x-1}$$

that is, formula (14).

To prove the identity (15), we notice that we generally have:

$$\mathcal{B}_{n+1}^{k}-\mathcal{B}_{n}^{k}=\frac{(x^{2n+2}-1)(x^{2n+1}-1)...(x^{2n+2-k}-1)}{(x-1)(x^{2}-1)...(x^{k+1}-1)}-\frac{(x^{2n}-1)(x^{2n-1}-1)...(x^{2n-k}-1)}{(x-1)(x^{2}-1)...(x^{k+1}-1)}=$$
$$=\frac{(x^{2n}-1)(x^{2n-1}-1)...(x^{2n+2-k}-1)}{(x-1)(x^{2}-1)...(x^{k+1}-1)}[(x^{2n+2}-1)(x^{2n+1}-1)-(x^{2n+1-k}-1)(x^{2n-k}-1)]$$

But

$$\left(x^{2n+2}-1\right)\left(x^{2n+1}-1\right)-\left(x^{2n+1-k}-1\right)\left(x^{2n-k}-1\right)=x^{4n+3}-x^{2n+2}-x^{2n-1}-x^{4n-1-2k}+$$
$$+x^{2n+1-k}+x^{2n-k}=x^{4n-2k+1}(x^{2k+2}-1)-x^{2n-k}\left(x^{k+1}-1\right)-x^{2n+1-k}\left(x^{k+1}-1\right)=$$
$$=\left(x^{k+1}-1\right)\left[x^{4n-2k+1}(x^{k+1}+1)-x^{2n-k}-x^{2n+1-k}\right]=(x^{k+1}-1)[x^{2n+1}(x^{2n-k+1}-1)+$$
$$+x^{2n-k+1}(x^{k}-1)+x^{2n-k}(x^{2n-k+1}-1)]$$

so that:

$$\mathcal{B}_{n+1}^{k} - \mathcal{B}_{n}^{k} = x^{2n+1}\frac{(x^{2n}-1)(x^{2n-1}-1)...(x^{2n-k+1}-1)}{(x-1)(x^{2}-1)...(x^{k}-1)} +$$
$$+x^{2n-k+1}\frac{(x^{2n}-1)(x^{2n-1}-1)...(x^{2n-k+1}-1)}{(x-1)(x^{2}-1)...(x^{k-1}-1)} + x^{2n-k}\frac{(x^{2n}-1)(x^{2n-1}-1)...(x^{2n-k+1}-1)}{(x-1)(x^{2}-1)...(x^{k}-1)}$$

or

$$\mathcal{B}_{n+1}^{k} - \mathcal{B}_{n}^{k} = x^{2n+1}\mathcal{B}_{n}^{k-1} + x^{2n-k+1}\mathcal{B}_{n}^{k-2} + x^{2n-k}\mathcal{B}_{n}^{k-1}$$

which means we have the formula (15).

We also have:

$$\mathcal{B}_{n+1}^{2n} = \frac{(x^{2n+2}-1)(x^{2n+1}-1)...(x^{2}-1)}{(x-1)(x^{2}-1)...(x^{2n+1}-1)} = \frac{x^{2n+2}-1}{x-1} = \frac{x^{2n+1}(x-1)+x^{2n+1}-1}{x-1}$$

or:

$$\mathcal{B}_{n+1}^{2n} = x^{2n+1} + x\frac{x^{2n}-1}{x-1} + 1$$

namely:

$$\mathcal{B}_{n+1}^{2n} = x^{2n+1}\mathcal{B}_{n}^{2n-1} + x\mathcal{B}_{n}^{2n-2} + \mathcal{B}_{n}^{2n-1}$$

so that formula (16) is demonstrated.

Finally, we have the formula (17).

Returning to formula (12) and taking into account formulas (13) - (17), we can write:

$$\mathcal{A}_{n} - \mathcal{A}_{n+1} = x^{2n+1} + x^{2n} - x^{2n+1}\mathcal{B}_{n}^{0} - x^{2n} - x^{2n-1}\mathcal{B}_{0} + x^{2n+1}\mathcal{B}_{n}^{1} - x^{2n-1}\mathcal{B}_{n}^{0} +$$
$$+x^{2n-2}\mathcal{B}_{n}^{1} - x^{2n+1}\mathcal{B}_{n}^{2} - x^{2n-2}\mathcal{B}_{n}^{1} + x^{2n-3}\mathcal{B}_{n}^{2} + ... +$$
$$+(-1)^{k}x^{2n+1}\mathcal{B}_{n}^{k-1} + (-1)^{k}x^{2n-k+1}\mathcal{B}_{n}^{k-2} + (-1)^{k}x^{2n-k}\mathcal{B}_{n}^{k-1} + ... -$$
$$-x^{2n+1}\mathcal{B}_{n}^{2n-2} - x^{2}\mathcal{B}_{n}^{2n-3} - x\mathcal{B}_{n}^{2n-2} + x^{2n+1}\mathcal{B}_{n}^{2n-1} + x\mathcal{B}_{n}^{2n-2} + \mathcal{B}_{n}^{2n-1} - \mathcal{B}_{n}^{2n-1}$$

We notice that the terms written in the second column are reduced to those written in the third column and we remain to:

$$\mathcal{A}_{n} - \mathcal{A}_{n+1} = x^{2n+1}(1 - \mathcal{B}_{n}^{0} + \mathcal{B}_{n}^{1} - \mathcal{B}_{n}^{2} + ... + (-1)^{k}\mathcal{B}_{n}^{k-1} + ... + \mathcal{B}_{n}^{2n-1})$$

or, according to formula (10), the large parenthesis is equal to $\mathcal{A}_{n}$ and we can write:

$$\mathcal{A}_{n} - \mathcal{A}_{n+1} = x^{2n+1}\mathcal{A}_{n}$$

or:

$$\mathcal{A}_{n+1} = (1 - x^{2n+1})\mathcal{A}_n$$

Replacing $\mathcal{A}_n$ with the second member of identity (8) that we assumed as valid for index n, we deduce that:

$$\mathcal{A}_{n+1} = (1-x)(1-x^3)...(1-x^{2n-1})(1-x^{2n+1}) \tag{19}$$

The expression of $\mathcal{A}_{n+1}$, (19) deduced from formula (8), which we assumed valid for the index n with $n+1$, it turns out that identity (8) is true whatever the natural number n . [2]

3. THE METHOD OF NON-DETERMINANT COEFFICIENTS AND RECURRENCE RELATIONS

As an example where this method is applied, let's deal with the following problem:
To find the law of product development:

$$\mathcal{P} = (x+t)(x+t^2)(x+t^3)...(x+t^n) \tag{20}$$

after the power of x .

It is obvious that $\mathcal{P}_n$ is a polynomial of degree n , whose first term is x^n . Let's write:

$$\mathcal{P}_n = x^n + \mathcal{A}_{n,1}x^{n-1} + \mathcal{A}_{n,2}x^{n-2} + ... + \mathcal{A}_{n,p}x^{n-p} + ... + \mathcal{A}_{n,n} \tag{21}$$

where the coefficients were noted $x^{n-1}, x^{n-2}, ..., 1$ cu $\mathcal{A}_{n,1}, \mathcal{A}_{n,2}, ..., \mathcal{A}_{n,n}$ first index n , indicating that they are the coefficients of the polynomial $\mathcal{P}_n$.

To determine the coefficients $\mathcal{A}_{n,1}, \mathcal{A}_{n,2}, ..., \mathcal{A}_{n,n}$ let us use the observation that:

$$\mathcal{P}_{n+1} = \mathcal{P}_n(x + t^{n+1})$$

where $\mathcal{P}_{n+1}$ is a polynomial of degree $n+1$, obtained from formula (21) by replacing the index n with $n+1$. There follows from the previous identity that we will have:

$$\begin{aligned} & x^{n+1} + \mathcal{A}_{n+1,1}\, x^{n} + \mathcal{A}_{n+1,2}\, x^{n-1} + ... + \mathcal{A}_{n+1,p}\, x^{n+1-p} + ... + \mathcal{A}_{n+1,n+1} = \\ & = x^{n+1} + \mathcal{A}_{n,1}\, x^{n} + \mathcal{A}_{n,2}\, x^{n-1} + ... + + \mathcal{A}_{n,p}\, x^{n+1-p} + ... + \mathcal{A}_{n,n}\, x + t^{n+1}x^n + \\ & + \mathcal{A}_{n,1}\, t^{n+1}x^{n-1} + ... + \mathcal{A}_{n,p-1}\, t^{n+1}x^{n+1-p} + ... + \mathcal{A}_{n,n-1}t^{n+1}x + \mathcal{A}_{n,n}t^{n+1} \end{aligned}$$

Assuming that the coefficients of the polynomial of the first member are equal to the corresponding coefficients of the second member, we will have the recurrence equations:

$$\begin{aligned}
&\mathcal{A}_{n+1,1} = \mathcal{A}_{n,1} + t^{n+1}\\
&\mathcal{A}_{n+1,2} = \mathcal{A}_{n,2} + t^{n+1}\mathcal{A}_{n,1}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\mathcal{A}_{n+1,p} = \mathcal{A}_{n,p} + t^{n+1}\mathcal{A}_{n,p-1}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\mathcal{A}_{n+1,n} = \mathcal{A}_{n,n} + t^{n+1}\mathcal{A}_{n,n-1}\\
&\mathcal{A}_{n+1,n+1} = t^{n+1}\mathcal{A}_{n,n}
\end{aligned} \tag{22}$$

which determines the coefficients $\mathcal{A}_{n,1}, \mathcal{A}_{n,2}, \ldots, \mathcal{A}_{n,n}$.

Let us first note from formulas (20) and (21) that the term $\mathcal{A}_{n,n}$ is determined immediately.

We have:

$$\mathcal{A}_{n,n} = t \cdot t^2 \cdot \ldots \cdot t^n = t^{\frac{n(n+1)}{2}} \tag{23}$$

With this in mind, we can move on to solving each recurrence equation (22). Let's consider the first one:

$$\mathcal{A}_{n+1,1} = \mathcal{A}_{n,1} + t^{n+1}$$

Considering that $\mathcal{A}_{1,1} = t$ and giving to n values $1, 2, \ldots, n-1$ we will have the formulas:

$$\begin{aligned}
&\mathcal{A}_{1,1} = t\\
&\mathcal{A}_{2,1} = \mathcal{A}_{1,1} + t^2\\
&\mathcal{A}_{3,1} = \mathcal{A}_{2,1} + t^3\\
&\ldots\ldots\ldots\ldots\\
&\mathcal{A}_{n,n} = \mathcal{A}_{n-1,1} + t^n
\end{aligned}$$

By summing all these formulas, we obtain:

$$\mathcal{A}_{n,1} = t + t^2 + \ldots + t^n \tag{24}$$

Taking into account the identity:

$$1+t+...+t^{n-1}=\frac{t^n-1}{t-1}$$

we can write more closely the previous formula in the form:

$$\mathcal{A}_{n,1}=t\cdot\frac{t^n-1}{t-1} \tag{25}$$

Let us now turn to the second recurrence equation (22), which according to formula (25) is written as:

$$\mathcal{A}_{n,+1,2}=\mathcal{A}_{n,2}+t^{n+2}\cdot\frac{t^n-1}{t-1}$$

Let us consider that $\mathcal{A}_{2,2}=t^3$ and give to n the values $2,3,...,n-1$. We'll get the formulas:

$$\mathcal{A}_{2,2}=t^3\cdot\frac{t-1}{t-1}$$

$$\mathcal{A}_{3,2}=\mathcal{A}_{2,2}+t^4\cdot\frac{t^2-1}{t-1}$$

$$\mathcal{A}_{4,2}=\mathcal{A}_{3,2}+t^5\cdot\frac{t^3-1}{t-1}$$

....

$$\mathcal{A}_{n,2}=\mathcal{A}_{n-1,2}+t^{n+1}\cdot\frac{t^{n-1}-1}{t-1}$$

Adding all these formulas, we deduce that:

$$\mathcal{A}_{n,2}=\frac{t^3}{t-1}\left[t(1+t^2+...+t^{2(n-2)})-(1+t+...+t^{n-2})\right]=\frac{t^3}{t-1}\left[\frac{t(t^{2n-2}-1)}{t^2-1}-\frac{t^{n-1}-1}{t-1}\right]$$

namely:

$$\mathcal{A}_{n,2}=t^3\,\frac{(t^n-1)(t^{n-1}-1)}{(t-1)(t^2-1)} \tag{26}$$

We leave it to the reader to show that:

$$\mathcal{A}_{n,3}=t^6\,\frac{(t^n-1)(t^{n-1}-1)(t^{n-2}-1)}{(t-1)(t^2-1)(t^3-1)} \tag{27}$$

and that, generally:

$$\mathcal{A}_{n,p}=t^{\frac{p(p+1)}{2}}\,\frac{(t^{\mathrm{n}}-1)(t^{\mathrm{n-1}}-1)...(t^{\mathrm{n-p+1}}-1)}{(t-1)(t^2-1)...(t^{\mathrm{p}}-1)} \tag{28}$$

for $p=4,5,...,n$

This will lead to the required development:

$$(x+t)(x+t^2)...(x+t^n) = x^n + t\frac{t^n-1}{t-1}x^{n-1} + t^3\frac{(t^n-1)(t^{n-1}-1)}{(t-1)(t^2-1)}x^{n-2} +$$
$$+ .... + \quad (29)\ [3]$$
$$+t^{\frac{p(p+1)}{2}}\frac{(t^n-1)(t^{n-1}-1)...(t^{n-p+1}-1)}{(t-1)(t^2-1)...(t^p-1)} + t^{\frac{n(n+1)}{2}}\frac{(t^n-1)(t^{n-1}-1)...(t-1)}{(t-1)(t^2-1)...(t^n-1)}$$

4. NEWTON'S BINOMIAL

As an application of identity (29), let us deduce *Newton's formula of identity*. For this we give a new form of identity (29), passing suddenly to the general term with the coefficient $\mathcal{A}_{n,p}$ given by formula (25), and dividing the numerator and denominator by $(t-1)^p$. We can write:

$$\mathcal{A}_{n,p} = t^{\frac{p(p+1)}{2}} \frac{\dfrac{(t^n-1)(t^{n-1}-1)...(t^{n-p+1}-1)}{t-1}}{\dfrac{(t-1)(t^2-1)...(t^p-1)}{t-1}} \qquad (30)$$

If we take identity into account that:

$$\frac{t^q-1}{t-1} = t^{q-1} + t^{q-2} + ... + 1$$

the previous expression of $\mathcal{A}_{n,p}$ is written in the form:

$$\mathcal{A}_{n,p} = t^{\frac{p(p+1)}{2}} \frac{(t^{n-1}+t^{n-2}+...+1)(t^{n-2}+t^{n-3}+...+1)...(t^{n-p}+t^{n-p-1}+...+1)}{1\cdot(t+1)(t^2+t+1)...(t^{p-1}+t^{p-2}+...+1)} \qquad (31)$$

By making $t=1$ in formula (31), we obtain:

$$\mathcal{A}_{n,p} = \frac{n(n-1)...(n-p+1-1)}{1\cdot 2\cdot ...\cdot p} \qquad (32)$$

and formula (19) becomes *the formula of Newton's binomial:*

$$(x+1)^n = x^n + \frac{n}{1}x^{n-1} + \frac{n(n-1)}{1\cdot 2}x^{n-2} + ... + \frac{n(n-1)\cdot ...\cdot 1}{1\cdot 2\cdot ...\cdot n} \qquad (33)\ [2].$$

R E F E R E N C E S

1. UDRIŞTE Constantin (coord.) *Culegere de probleme de matematică*. Enunţuri, The Didactic and Paedagogic Publishing House, Bucharest, Romania, 1989.
2. IONESCU D.V., *Complemente de matematici pentru licee*, The Didactic and Paedagogic Publishing House, Bucharest, Romania, 1987.
3. RIZESCU Gheorghe, RIZESCU Eugenia, *Teme pentru cercurile de matematică din licee*, vols. I - II, The Didactic and Pa edagogic Publishing House, Bucharest, Romania 1977.