Memoirs of the Scientific Sections of the Romanian Academy
Tome XLII, 2019

MATHEMATICS

ASSOCIATED METRICS FOR BOURGEOIS' CONTACT FORMS

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1. INTRODUCTION

In 2002, Frédéric Bourgeois [2] showed that, given a compact contact manifold M^{2n-1}, the product $M^{2n-1} \times T^2$ also carries a contact form; then, as a corollary he observed that all odd-dimensional tori have contact forms. The idea is to use an open book decomposition of M^{2n-1} that is compatible with its contact structure [4] to produce contact forms on the product $M^{2n-1} \times T^2$. Here we first find the Reeb vector field and then a method for constructing associated metrics for contact forms of the type studied by Bourgeois. We will also discuss $S^2 \times T^2$ in some detail. While the procedure would apply to tori, the construction is quite difficult and we will make only some remarks.

Keywords: mathematical modeling, applied mathematics, Nicole Oresme, 14th century.

2. CONTACT METRIC MANIFOLDS

By a *contact manifold* we mean a C^∞ manifold M^{2n+1} together with a 1-form η so that

$$\eta \wedge (d\eta)^n \neq 0.$$

It is well known that, given η, there exists a unique vector field Z_η so that $d\eta(X, Z_n) = 0$ and $\eta(Z_\eta) = 1$. The vector field Z_η is known as the *Reeb vector field* of the contact form η. Denote by $\mathcal{D}_\eta$ the *contact sub-bundle* defined by:

$$\{X \in T_m M : \eta(X) = 0\}.$$

Roughly speaking, the meaning of the contact condition $\eta \wedge (d\eta)^n \neq 0$ is that the contact subbundle is as far from being integrable as possible or, thinking about it another way, the contact subbundle rotates as one moves around on the manifold. In fact, the maximum dimension of an integral submanifold of $\mathcal{D}_\eta$ is only n.

The simplest contact manifold is $\mathbb{R}^{2n+1}(x^1,...,x^n,y^1,...,y^n,z)$ with the form:

$$\eta = dz - \sum_{i=1}^{n} y^i dx^i$$

or $\eta = dt - \sum_{i=1}^{n} p^i dq^i$ if thinking in terms of classical mechanics. The classical theorem of Darboux states that, given any contact form η, one can always find local coordinates so that n takes this form.

A Riemannian metric g is an *associated metric* for a contact form η if, first of all:

$$\eta(X) = g(X, Z_\eta)$$

and secondly, there exists a field of endomorphisms ϕ so that:

$$\phi^2 = -1 + \eta \otimes Z_n \quad \text{and} \quad d\eta(X,Y) = g(X,\phi Y).$$

In particular, the contact sub-bundle is orthogonal to the Reeb vector field, $\phi Z_n = 0$, $\eta \circ \phi = 0$ and ϕ acts as an almost complex structure on $\mathcal{D}_\eta$. We refer to (ϕ, Z_η, η, g) as a *contact metric structure* and to M^{2n+1} with such a structure as a *contact metric manifold.*

Another important structure tensor is the Lie derivative of Φ with respect to the Reeb vector field; we set $h = \frac{1}{2}£_{Z_\eta}\Phi$. The operator h is symmetric, anti-commutes with Φ, $hZ_n = 0$ and h vanishes if and only if the Reeb vector field is Killing. An important property of h is the following:

$$\nabla_X Z_\eta = -\phi X - \phi h X$$

which reflects the rotation of the Reeb vector field and in turn, by orthogonality, of the contact subbundle. An immediate consequence is that the integral curves of Z_η are geodesics. We also observe that, if λ is a non-zero eigenvalue of h with eigenvector X, then $-\lambda$ is an eigenvalue with eigenvector ϕX. Related to the Ricci curvature in the direction of the Reeb vector field, we have the following important result on a contact metric manifold M^{2n+1}:

$$Ric(Z_\eta) = 2n - \mathrm{tr}h^2. \qquad (*)$$

Associated metrics can be constructed in the following manner. Let $\bar{\bar{g}}$ be any Riemannian metric on M^{2n+1} and define a metric $\bar{g}$ by:

$$\bar{g}(X,Y) = \bar{\bar{g}}(-X + \eta(X)Z_\eta, -Y + \eta(Y)Z_\eta) + \eta(X)\eta(Y).$$

Then $\bar{g}(X, Z_\eta) = \eta(X)$. Now, choose a local $\bar{g}$-orthonormal basis $\{X_1, ..., X_{2n}\}$ of $\mathcal{D}_\eta$ and evaluate $d\eta$ on these vectors. This gives a $2n \times 2n$ non-singular matrix, $A_{ij} = d\eta(X_i, X_j)$ which, by polarization, can be written as the product of an orthogonal matrix F and a positive definite symmetric matrix G. Define an associated metric g and an almost complex structure ϕ on $\mathcal{D}_\eta$ by $g(X_i, X_j) = G_{ij}$ and $\phi X_i = F_i^j X_j$ and extend it to all tangent vectors by $g(X, Z_\eta) = \eta(X)$ and $\phi Z_\eta = 0$. If $\{Y_1, ..., Y_{2n}\}$ is another $\bar{g}$-orthonormal basis of $\mathcal{D}_\eta$ on an overlapping neighborhood, one can show by the uniqueness of the polar decomposition that g and ϕ are globally defined.

Since the metric $\bar{\bar{g}}$ is totally arbitrary, the procedure can become overly complicated. A variation of this idea is that, if the form is presented explicitly enough, one might try to construct local vector fields $\{X_1, ..., X_{2n}\}$ by spanning $\mathcal{D}_\eta$ over a neighborhood $\mathcal{U}$, so that matrix $d\eta = \left(X_i, X_j\right)$ is relatively simple to polarize. Then, in the above procedure, define the initial metric by declaring $\{X_1, ..., X_{2n}\}$ together with Z_η to be orthonormal, and use this as $\bar{g}$, giving an associated metric $\bar{g}$ locally. Then $\bar{g}$ can be extended and the above procedure used to give a global associated metric.

The space of all associated metrics for a given contact form is infinitely dimensional, so that the associated metrics are far from being unique. All associated metrics have the same volume element, proportional to $\eta \wedge (d\eta)^n$.

A general reference for the ideas of this section is [1]; see also p.112 for a proof of equation $(*)$.

3. THE BOURGEOIS CONSTRUCTION

As already observed, Bourgeois' goal is to construct a contact form on the product $M^{2n-1} \times T^2$, where M^{2n-1} is a compact contact manifold. Then, since T^3 has a well-known contact structure, all odd-dimensional tori have contact structures. In particular, we have the following theorem.

Theorem [2]. Let M^{2n-1} be a closed contact manifold. Then the product $M^{2n-1} \times T^2$ admits a contact structure that is invariant under the natural T^2-action.

Bourgeois begins with Giroux's idea [4] of an open book decomposition of a manifold M compatible with a given contact form α on M. In particular, one has a co-dimension 2 submanifold N called the *binding,* a smooth, locally trivial fibration $\psi : M \backslash N \to S^1$ whose fibres $\psi^{-1}(\theta)$, $\theta \in S^1$ are called the *pages,* and N

has a tubular neighborhood $N \times D^2$, in which ψ is given by the angular coordinate in $\mathcal{D}^2$. This open book decomposition is said to be *compatible* with a contact form α; if the restriction of α to N is a contact form, $d\alpha$ is symplectic on the pages and the orientation induced on N by α is compatible with that of the symplectic structure on the pages. Note that the Reeb vector field of α is tangent to N. Giroux and Mohsen then prove the following theorem.

Theorem [4]. Given a contact form on a manifold, there exists a C^0-close contact form α and a compatible open book decomposition.

Now consider a contact manifold M^{2n-1} with contact form α, and a compatible open book decomposition with binding N. For an arbitrary metric, let $\rho : M^{2n-1} \to \mathbb{R}$ denote the distance function to N while, for fixed ε let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth increasing function so that $f(x) = x$ if $0 \le x \le \varepsilon$ and $f(x) = (3/2)\varepsilon$ if $x \ge 2\varepsilon$. Multiplying ψ and $f \circ \rho$, we obtain smooth functions ψ_1 and ψ_2. Now, on the manifold $M^{2n-1} \times T^2$ define a contact form η by:

$$\eta = \alpha + \frac{\psi_1}{2} d\theta_{2n} + \frac{\psi_2}{2} d\theta_{2n+1}.$$

Bourgeois then computes $\eta \wedge (d\eta)^n$, obtaining:

$$\eta \wedge (d\eta)^n = \frac{n}{4}(d\alpha)^{n-1} \wedge (\psi_2 d\psi_1 - \psi_1 d\psi_2) \wedge d\theta_{2n} \wedge d\theta_{2n+1}$$
$$- \frac{n(n-1)}{4} \alpha \wedge (d\alpha)^{n-2} \wedge d\psi_1 \wedge d\psi_2 \wedge d\theta_{2n} \wedge d\theta_{2n+1}.$$

Since the symplectic form $d\alpha$ on the pages is compatible with the orientation, the first summand is a positive volume form away from $N \times T^2$. Since α is a positive contact form on N, the second summand is non-negative on M^{2n+1} and positive near $N \times T^2$. See also [3] and sections 4.4.2 and 7.3 for further discussion.

4. THE REEB VECTOR FIELD FOR A BOURGEOIS CONTACT FORM

On $M^{2n-1} \times T^2$ consider a local basis of the form:

$$X_1, X_2, \ldots, X_{2n-3}, X_{2n-2}, Z_\alpha, \partial_{2n} := \frac{\partial}{\partial\theta_{2n}}, \partial_{2n+1} := \frac{\partial}{\partial\theta_{2n+1}}$$

where the X_i's span $\mathcal{D}_\alpha$. Moreover this basis can be chosen so that:

$$d\alpha(X_{2i-1},X_{2i})=-1,\quad i=1,...,n-1$$

where for other pairings of the basis vectors da is zero. The Reeb vector field will take the form:

$$Z_\eta = Z^1X_1 + Z^2X_2 + ...Z^{2n-2}X_{2n-2} + CZ_\alpha + D\partial_{2n} + E\partial_{2n+1}.$$

Now, $d\eta(X,Z_\eta)=0$ for all X and $\eta(Z_\eta)=1$, and we can solve it for the coefficients. Setting:

$$\Delta=\sum_{i=1}^{n-1} d\psi_1\wedge d\psi_2(X_{2i-1},X_{2i}) \quad \text{and} \quad \Theta=(\psi_2 d\psi_1-\psi_1 d\psi_2)(Z_\alpha)$$

we have:

$$Z^{2i-1}=\frac{d\psi_1\wedge d\psi_2(X_{2i},Z_\alpha)}{\Delta+\Theta},\quad Z^{2i}=-\frac{d\psi_1\wedge d\psi_2(X_{2i-1},Z_\alpha)}{\Delta+\Theta},$$

$$C=\frac{\Delta}{\Delta+\Theta},\quad D=\frac{-2(Z_\alpha\psi_2)}{\Delta+\Theta}\quad\text{and}\quad E=\frac{2(Z_\alpha\psi_1)}{\Delta+\Theta}$$

which gives coefficients of Z_η.

5. THE ASSOCIATED METRICS

Having found the Reeb vector field Z_η for the contact form, we introduce vector fields with respect to which $d\eta$ takes on a canonical form. Let:

$$Y_1=\frac{1}{\sqrt{2|\Delta+\Theta|}}\left\{-\sum_{i=1}^{n-1}(X_{2i}\psi_1)X_{2i-1}+\sum_{i=1}^{n-1}(X_{2i-1}\psi_1)X_{2i}-2\psi_1 Z_\alpha+4\partial_{2n}\right\},$$

$$Y_2=\frac{\varepsilon}{\sqrt{2|\Delta+\Theta|}}\left\{\sum_{i=1}^{n-1}(X_{2i}\psi_2)X_{2i-1}-\sum_{i=1}^{n-1}(X_{2i-1}\psi_2)X_{2i}+2\psi_2 Z_\alpha-4\partial_{2n+1}\right\}$$

where $\varepsilon=\mathrm{sgn}(\Delta+\Theta)$. Then, $Y_1,Y_2\in\mathcal{D}_\eta$ and, in addition to $d\eta(X_{2i-1},X_{2i})=-1$, we obtain by direct computation $d\eta(X_r,Y_1)=d\eta(X_r,Y_2)=0$, $r=1,...,2n-2$, and $d\eta(Y_1,Y_2)=-1$.

Thus, the matrix of $d\eta|\mathcal{D}_\eta$ with respect to these vector fields is the matrix with n diagonal blocks of the form:

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

and zeros elsewhere. Polarizing is just recognizing that an associated metric is given by the identity regarding Z_η as unit and orthogonal to $\mathcal{D}_\eta$.

To make this more meaningful, we return to the original basis:

$$X_1, X_2, \ldots, X_{2n-3}, X_{2n-2}, Z_\alpha, \partial_{2n}, \partial_{2n+1}.$$

Solving for $Z_\alpha, \partial_{2n}, \partial_{2n+1}$ we have:

$$Z_\alpha = Z_\eta + \frac{1}{\sqrt{2|\Delta+\Theta|}}(\varepsilon(Z_\alpha\psi_2)Y_1 + (Z_\alpha\psi_1)Y_2),$$

$$\partial_{2n} = \frac{1}{4}\left(\sum_{i=1}^{n-1}(X_{2i}\psi_1)X_{2i-1} - \sum_{i=1}^{n-1}(X_{2i-1}\psi_1)X_{2i} + 2\psi_1 Z_\alpha + \sqrt{2|\Delta+\Theta|}Y_1\right),$$

$$\partial_{2n+1} = \frac{1}{4}\left(\sum_{i=1}^{n-1}(X_{2i}\psi_2)X_{2i-1} - \sum_{i=1}^{n-1}(X_{2i-1}\psi_2)X_{2i} + 2\psi_2 Z_\alpha + \varepsilon\sqrt{2|\Delta+\Theta|}Y_2\right).$$

For the sake of simplicity, set $P = 1 + \dfrac{(Z_\alpha\psi_1)^2 + (Z_\alpha\psi_2)^2}{2|\Delta+\Theta|}$. It is then straightforward to compute the components of the associated metric g with respect to the original basis.

Theorem. Let a a contact form on the manifold M^{2n-1} with a compatible open book decomposition. Then, for the contact form $\eta = \alpha + \dfrac{\psi_1}{2}d\theta_{2n} + \dfrac{\psi_2}{2}d\theta_{2n+1}$ on $M^{2n-1}\times T^2$, one has an associated metric g given by $g(X_r, X_S) = \delta_{rS}$,

$$g(X_{2i-1}, Z_\alpha) = 0, \quad g(X_{2i-1}, \partial_{2n}) = \frac{1}{4}X_{2i}\psi_1, \quad g(X_{2i-1}, \partial_{2n+1}) = \frac{1}{4}X_{2i}\psi_2,$$

$$g(X_{2i}, Z_\alpha) = 0, \quad g(X_{2i}, \partial_{2n}) = -\frac{1}{4}X_{2i-1}\psi_1, \quad g(X_{2i}, \partial_{2n+1}) = -\frac{1}{4}X_{2i-1}\psi_2,$$

$$g(Z_\alpha, Z_\alpha) = P, \quad g(Z_\alpha, \partial_{2n}) = \frac{\psi_1}{2}P + \frac{\varepsilon}{4}Z_\alpha\psi_2, \quad g(Z_\alpha, \partial_{2n+1}) = \frac{\psi_2}{2}P - \frac{\varepsilon}{4}Z_\alpha\psi_1,$$

$$g(\partial_{2n}, \partial_{2n}) = \frac{1}{16}\sum_{r=1}^{2n-2}(X_r\psi_1)^2 + \frac{1}{4}\psi_1^2 P + \frac{\varepsilon}{4}\psi_1 Z_\alpha\psi_2 + \frac{1}{8}|\Delta+\Theta|,$$

$$g(\partial_{2n}, \partial_{2n+1}) = \frac{1}{16}\sum_{r=1}^{2n-2}(X_r\psi_1)(X_r\psi_2) + \frac{1}{4}\psi_1\psi_2 P - \frac{\varepsilon}{8}\psi_1 Z_\alpha\psi_1 + \frac{\varepsilon}{8}\psi_2 Z_\alpha\psi_2,$$

$$g(\partial_{2n+1}, \partial_{2n+1}) = \frac{1}{16}\sum_{r=1}^{2n-2}(X_r\psi_2)^2 + \frac{1}{4}\psi_2^2 P - \frac{\varepsilon}{4}\psi_2 Z_\alpha\psi_1 + \frac{1}{8}|\Delta+\Theta|.$$

For use in Section 6, we exhibit the matrix of g in the 5-dimensional case.

$$g=\begin{pmatrix} 1 & 0 & 0 & \frac{X_2\psi_1}{4} & \frac{X_2\psi_2}{4} \\ 0 & 1 & 0 & -\frac{X_1\psi_1}{4} & -\frac{X_1\psi_2}{4} \\ 0 & 0 & P & \frac{\psi_1}{2}P+\frac{\varepsilon Z_\alpha\psi_2}{4} & \frac{\psi_2}{2}P-\frac{\varepsilon Z_\alpha\psi_1}{4} \\ \frac{X_2\psi_1}{4} & -\frac{X_1\psi_1}{4} & \frac{\psi_1}{2}P+\frac{\varepsilon Z_\alpha\psi_2}{4} & g(\partial_4.\partial_4) & g(\partial_4.\partial_5) \\ \frac{X_2\psi_2}{4} & -\frac{X_1\psi_2}{4} & \frac{\psi_2}{2}P-\frac{\varepsilon Z_\alpha\psi_1}{4} & g(\partial_4.\partial_5) & g(\partial_5.\partial_5) \end{pmatrix}$$

Note also that $\phi X_{2i-1}=X_{2i}$, $\phi X_{2i}=-X_{2i-1}$, $\phi Y_1=Y_2$, $\phi Y_2=-Y_1$ and

$$\phi Z_\alpha=\frac{1}{\sqrt{2|\Delta-\Theta|}}(\varepsilon(Z_\alpha\psi_2)Y_2-(Z_\alpha\psi_1)Y_1);$$

then one can compute ϕ in terms of

$$X_1,X_2,\ldots,X_{2n-3},X_{2n-2},Z_\alpha,\partial_{2n},\partial_{2n+1}$$

or in terms of coordinates.

6. EXAMPLES AND REMARKS

As an example, let us consider the well-known 3-sphere with its standard Sasakian structure inherited as the unit sphere in $\mathbb{C}^2$ with the usual complex structure. The almost complex structure acting on the outer normal gives the Reeb vector field of the Sasakian structure. Stereographic projecting to $\mathbb{R}^3$ with cylindrical coordinates (r,θ,z) gives the contact form:

$$\alpha=\frac{2}{(1+r^2+z^2)^2}(2rzdr+2r^2d\theta+(1-r^2+z^2)dz).$$

The Reeb vector field is:

$$Z_\alpha=rz\partial_r+\partial_\theta+\frac{1}{2}(1-r^2+z^2)\partial_z$$

and

$$\alpha\wedge d\alpha=\frac{16r}{(1+r^2+z^2)^3}dr\wedge d\theta\wedge dz.$$

The vector fields:

$$X_1 = -\frac{1}{2}(1+r^2-z^2)\partial_r + \frac{z}{r}\partial_\theta - rz\partial_z$$

$$X_2 = z\partial_r + \frac{1}{2r}(1-r^2-z^2)\partial_\theta - r\partial_z$$

belong to $\mathcal{D}_\alpha$ and satisfy $d\alpha(X_1, X_2) = -1$.

We now consider the contact form:

$$\eta = \frac{2}{(1+r^2+z^2)^2}\left(2rzdr + 2r^2d\theta + (1-r^2+z^2)dz\right) + \frac{1}{2}\cos\theta d\theta_4 + \frac{1}{2}\sin\theta d\theta_5$$

On $S^3 \times T^2$; here we have:

$$\eta \wedge (d\eta)^2 = \frac{-8r}{(1+r^2+z^2)^3} dr \wedge d\theta \wedge dz \wedge d\theta_4 \wedge d\theta_5.$$

Differentiating $\psi_1 = \cos\theta$ and $\psi_2 = \sin\theta$, we have:

$$X_1\psi_1 = -\frac{z}{r}\sin\theta, \qquad X_1\psi_2 = \frac{z}{r}\cos\theta,$$

$$X_2\psi_1 = -\frac{1}{2r}(1-r^2-z^2)\sin\theta, \quad X_2\psi_2 = -\frac{1}{2r}(1-r^2-z^2)\cos\theta,$$

$$Z_\alpha\psi_1 = -\sin\theta, \qquad Z_\alpha\psi_2 = \cos\theta.$$

This gives:

$$\Delta = d\psi_1 \wedge \psi_2(X_1, X_2) = 0 \quad \text{and} \quad \Theta = \psi_2 Z_\alpha \psi_1 - \psi_1 Z_\alpha \psi_2 = -1.$$

The vector fields Y_1 and Y_2 become

$$Y_1 = \frac{1}{\sqrt{2}}\left\{\left(-\frac{1}{4r}(1+r^2+z^2)(1-r^2+z^2)(\sin\theta - 2rz\cos\theta)\right)\partial_r - 2\cos\theta\partial_\theta + \right.$$
$$\left. +\left(\frac{z}{2}(1+r^2+z^2)\sin\theta - (1-r^2+z^2)\cos\theta\right)\partial_z + 4\partial_4\right\},$$

$$Y_2 = \frac{1}{\sqrt{2}}\left\{\left(\frac{1}{4r}(1+r^2+z^2)(1-r^2+z^2)(\cos\theta - 2rz\sin\theta)\right)\partial_r - 2\sin\theta\partial_\theta + \right.$$
$$\left. +\left(-\frac{z}{2}(1+r^2+z^2)\cos\theta - (1-r^2+z^2)\sin\theta\right)\partial_z + 4\partial_5\right\}.$$

The Reeb vector field of η is:

$$Z_\eta = 2\cos\theta\partial_4 + 2\sin\theta\partial_5.$$

and the relation between Z_α and Z_η is:

$$Z_\alpha = Z_\eta - \frac{1}{\sqrt{2}}(\cos\theta Y_1 + \sin\theta Y_2).$$

Next note that for the associated metric g, as given in the Theorem,

$$g(Z_\alpha, Z_\alpha) = \frac{3}{2}, \quad g(Z_\alpha, Y_1) = -\frac{1}{\sqrt{2}}\cos\theta, \quad g(Z_\alpha, Y_2) = -\frac{1}{\sqrt{2}}\sin\theta.$$

the associated metric was computed with respect to coordinates $(r,\theta,z,\theta_4,\theta_5)$, which is quite complicated. For example:

$$g_{33} = \frac{4}{(1+r^2+z^2)^2} + \frac{2(1-r^2+z^2)^2}{(1+r^2+z^2)^4},$$

$$g_{34} = -\frac{z}{2(1+r^2+z^2)}\sin\theta + \frac{(1-r^2+z^2)}{(1+r^2+z^2)^2}\cos\theta,$$

$$g_{44} = \frac{(1+r^2+z^2)^2 - 4r^2}{64r^2}\sin^2\theta + \frac{1}{8}(1+\cos^2\theta).$$

A complete list of the components of this metric is attached as an appendix.

Similarly, the components of ϕ with respect to the coordinate basis are complicated; even if ϕ is a skew-symmetric operator, its matrix of components is not skew-symmetric (again, see the attached appendix). The subsequent computation of operator h is not difficult, so that one easily obtains that the non-zero eigenvalues of h are each with multiplicity 1. From the basic relation for the Ricci curvature in the direction of the Reeb vector field $(*)$.

$$\mathrm{Ric}(Z_\eta) = 4 - \mathrm{tr}h^2 = 4 - \frac{1}{8}(1 + \frac{(1+r^2+z^2)^2}{4r^2})^2$$

we note the change in sign of the curvature as one moves on $S^3 \times T^2$.

Ideally, one should next consider starting with the standard contact form on the 3-torus, viz. $\cos\theta_3 d\theta_1 + \sin\theta_3 d\theta_2$, even if this involves finding suitable functions ψ_1 and ψ_2. R. Lutz [5] gave a very explicit contact form on T^5, which provides the functions ψ_1 and ψ_2, however the α part of the Lutz form is not on global contact on T^3.

A simpler example, though not compact, is the cylinder $M^3 = S^1 \times \mathbb{R} \times S^1$ and the contact form on $M^3 \times T^2$ given by:

$$\eta = \frac{1}{2}(c_3 d\theta_1 + s_3 d\theta_2 + s_3 d\theta_4 + \theta_2 d\theta_5)$$

where we have set $c_3 = \cos\theta_3$ and $s_3 = \sin\theta_3$. Then:

$$\eta \wedge (d\eta)^2 = \frac{1}{4} d\theta_1 \wedge \ldots \wedge d\theta_5$$

therefore η is a global contact form. The Reeb vector field of η is $Z_\eta = 2(c_3\partial_1 + s_3\partial_4)$ whereas $Z_\alpha = 2(c_3\partial_1 + s_3\partial_2)$. Here, we take $X_1 = 2(-s_3\partial_1 + c_3\partial_2)$ and $X_2 = 2\partial_3$. The metric given in terms of $X_1, X_2, Z_\alpha, \partial_4, \partial_5$ becomes

$$g = \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 2c_3 & 0 \\ 0 & 4 & 0 & 0 & -2c_3 \\ 0 & 0 & 4(1+s_3^2) & 2s_3^3 & 2\theta_2(1+s_3^2) \\ 2c_3 & 0 & 2s_3^3 & 1+c_3^4 & \theta_2 s_3^3 \\ 0 & -2c_3 & 2\theta_2(1+s_3^2) & \theta_2 s_3^3 & 1+c_3^2+\theta_2^2(1+s_3^2) \end{pmatrix}$$

Computation of ϕ and h give as the non-zero eigenvalues of h each with multiplicity 1, and in turn, by $(*)$, we have $Ric(Z_\eta) = 2 - 4c_3^2 - 2c_3^4$.

With an eye to future research, one may try to abstract the above properties and other examples to give a new class of contact metric manifolds deserving of study. As a first remark in this direction, note that, in the two examples given above, operator h has exactly two non-zero eigenvalues. Similarly, one might consider the case where h has two zero eigenvalues with eigenvectors orthogonal to the Reeb vector field and the others non-zero. Of course, in dimension 5, they are the same. It is quite likely that one might also want to add another condition, possibly one involving the curvature.

REFERENCES

1. BLAIR D. E., *Riemannian Geometry of Contact and Symplectic Manifolds*, Second Edition, Birkhauser, Boston, 2010.
2. BOURGEOIS F., *Odd dimensional tori are contact manifolds*, International Mathematics Research Notices,2002, 30, 1571–1574.
3. GEIGES H., *An Introduction to Contact Topology*, Cambridge University Press, Cambridge, 2008.
4. GIROUX E., *Geometrie de contact: de la dimension trois vers les dimensions superieures*, Proceedings of the International Congress of Mathematicians, Vol. II, Higher Ed. Press, Beijing, 2002, 405–414.
5. LUTZ R., *Sur la geometrie des structures de contact invariantes*, Ann. Inst. Fourier, Grenoble, 1979, 29, 283–306.

APPENDIX

$$g_{11} = \frac{4}{(1+r^2+z^2)^2} + \frac{8r^2z^2}{(1+r^2+z^2)^4},$$

$$g_{12} = \frac{8r^3z}{(1+r^2+z^2)^4}, \quad g_{13} = \frac{4rz(1-r^2+z^2)}{(1+r^2+z^2)^4},$$

$$g_{14} = \frac{(1-r^2+z^2)}{4r(1+r^2+z^2)}\sin\theta + \frac{2rz}{(1+r^2+z^2)^2}\cos\theta,$$

$$g_{15} = \frac{-(1-r^2+z^2)}{4r(1+r^2+z^2)}\cos\theta + \frac{2rz}{(1+r^2+z^2)^2}\sin\theta,$$

$$g_{22} = \frac{4r^2}{(1+r^2+z^2)^2} + \frac{r^2}{(1+r^2+z^2)^4},$$

$$g_{23} = \frac{4r^2(1-r^2+z^2)}{(1+r^2+z^2)^4}, \quad g_{24} = \frac{2r^2}{(1+r^2+z^2)^2}\cos\theta,$$

$$g_{25} = \frac{2rz}{(1+r^2+z^2)^2}\sin\theta, \quad g_{33} = \frac{4}{(1+r^2+z^2)^2} + \frac{2(1-r^2+z^2)^2}{(1+r^2+z^2)^4},$$

$$g_{34} = \frac{-z}{2(1+r^2+z^2)}\sin\theta + \frac{(1-r^2+z^2)}{(1+r^2+z^2)^2}\cos\theta,$$

$$g_{35} = \frac{z}{2(1+r^2+z^2)}\cos\theta + \frac{(1-r^2+z^2)}{(1+r^2+z^2)^2}\sin\theta,$$

$$g_{44} = \frac{(1+r^2+z^2)^2-4r^2}{64r^2}\sin^2\theta + \frac{1}{8}(1+\cos^2\theta),$$

$$g_{45} = \frac{-((1+r^2+z^2)^2-4r^2)}{64r^2}\sin\theta\cos\theta + \frac{1}{8}\sin\theta\cos\theta,$$

$$g_{55} = \frac{(1+r^2+z^2)^2-4r^2}{64r^2}\cos^2\theta + \frac{1}{8}(1+\sin^2\theta),$$

$$\phi_1^1 = \frac{-z(1-r^2+z^2)}{2(1+r^2+z^2)}, \quad \phi_2^1 = \frac{r(1-r^2+z^2)}{2(1+r^2+z^2)},$$

$$\phi_3^1 = \frac{-(1+r^2+z^2)}{4r} - \frac{r(1-z^2)}{(1+r^2+z^2)}, \quad \phi_4^1 = \frac{-rz}{4}\sin\theta, \quad \phi_5^1 = \frac{rz}{4}\cos\theta,$$

$$\phi_1^2 = \frac{-(1-r^2+z^2)}{r(1+r^2+z^2)}, \qquad \phi_2^2 = 0, \qquad \phi_3^2 = \frac{2z}{(1+r^2+z^2)},$$

$$\phi_4^2 = \frac{-1}{4}\left(1+\frac{(1+r^2+z^2)^2}{4r^2}\right)\sin\theta, \quad \phi_5^2 = \frac{1}{4}\left(1+\frac{(1+r^2+z^2)^2}{4r^2}\right)\cos\theta,$$

$$\phi_1^3 = \frac{r(2+z^2)}{(1+r^2+z^2)}, \quad \phi_2^3 = \frac{-r^2 z}{(1+r^2+z^2)}, \quad \phi_3^3 = \frac{z(1-r^2+z^2)}{2(1+r^2+z^2)},$$

$$\phi_4^3 = \frac{-1}{8}(1-r^2+z^2)\sin\theta, \quad \phi_5^3 = \frac{1}{8}(1-r^2+z^2)\cos\theta,$$

$$\phi_1^4 = \frac{8rz}{(1+r^2+z^2)^2}\sin\theta, \quad \phi_2^4 = \frac{8r^2}{(1+r^2+z^2)^2}\sin\theta,$$

$$\phi_3^4 = \frac{4(1-r^2+z^2)}{(1+r^2+z^2)^2}\sin\theta, \qquad \phi_4^4 = \sin\theta\cos\theta, \quad \phi_5^4 = -\cos^2\theta,$$

$$\phi_1^5 = \frac{-8rz}{(1+r^2+z^2)^2}\cos\theta, \quad \phi_2^5 = \frac{8rz}{(1+r^2+z^2)^2}\cos\theta,$$

$$\phi_3^5 = \frac{-4(1-r^2+z^2)}{(1+r^2+z^2)^2}\cos\theta, \qquad \phi_4^5 = \sin^2 t, \quad \phi_5^5 = -\sin\theta\cos\theta,$$