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NONLINEAR SECOND-ORDER PARTIAL DIFFERENTIAL EQUATION-BASED IMAGE SMOOTHING TECHNIQUE

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A second-order nonlinear parabolic PDE-based restoration model is provided in this article. The proposed anisotropic diffusion-based denoising approach is based on some robust versions of the edge-stopping function and of the conductance parameter. Two stable and consistent approximation schemes are then developed for this differential model. Our PDE-based filtering technique achieves an efficient noise removal while preserving the edges and other image features. It outperforms both the conventional filters and also many PDE-based denoising approaches, as it results from the successful experiments and method comparison applied.

Keywords: Image denoising, second-order diffusion, diffusivity function, conductance parameter, numerical approximation scheme.

1. INTRODUCTION

The partial differential equation (PDE)-based models have been increasingly used in many traditional engineering areas, such as image processing and analysis, or computer vision, during the past three decades. The PDEs have been successfully used for solving numerous image processing and computer vision tasks. Many image processing and analysis techniques using variational and PDE-based algorithms have been developed recently, given the modeling flexibility and the advantages of the numerical implementation of PDEs (Guichard, 2001).

The detail-preserving image restoration problem still represents a focus in the image processing area, and a very big challenge for researchers. The conventional two dimensional denoising techniques, such as the averaging, Wiener and 2D Gaussian filters, can reduce the noise, but also have the disadvantage of blurring the edges and have no localization property (Jain, 1989). For this reason, a lot of edge-preserving PDE-based methods have been introduced in the last decades (Guichard, 2001, Weickert, 1998).

Several nonlinear anisotropic diffusion-based noise reduction models have been proposed since 1987, when the influential scheme of P. Perona and J. Malik was introduced (Perona, 1987). Since it is common to derive a PDE-based model from a variational problem, numerous variational restoration methods have been also constructed in the last decades (Weickert, 1998). The most influential variational image filtering algorithm is that developed by Rudin, Osher and Fetami in 1992 (Rudin, 1992). These variational and second-order PDE approaches overcome the blurring effect but often generate the undesired staircase effect (Buades, 2006).

In the present study, we propose a novel image denoising approach, using a nonlinear second-order anisotropic diffusion model, which alleviates the staircase effect and outperforms the state-of-the-art PDE-based methods (Guichard, 2001). The proposed PDE models are described in the following section, while two numerical approximation schemes are presented in the third one. The performed smoothing experiments and comparison method are described in the fourth section. The paper ends with a conclusion section, acknowledgements and a reference list.

2. NONLINEAR SECOND-ORDER ANISOTROPIC DIFFUSION SCHEME

We consider a novel nonlinear anisotropic diffusion-based model that provides an efficient noise removal while preserving successfully the image boundaries. The PDE-based denoising technique is based on the following second-order parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = div\left(\psi_u\left(\|\nabla u\|\right)\nabla u\right) - \nu\left(u - u_0\right), \left(x, y\right) \in \Omega \\ u\left(0, x, y\right) = u_0 \end{cases} \tag{1}$$

where u_0 is the degraded image, its domain $\Omega \subset R^2$ and $\nu \in (0,1)$. We construct the next diffusivity (edge-stopping) function $\psi_u : [0,\infty) \to [0,\infty)$, for this restoration scheme:

$$\psi_u(s) = \frac{\lambda}{\left(\dfrac{s}{K(u)}\right)^2 + K(u)\left|\log_{10}\left(\dfrac{s}{K(u)}\right)\right|} \tag{2}$$

where conductance diffusivity depends on the state of the evolving image at time *t*.

We consider a statistics-based automatic computation of this parameter, using the image noise estimation at each time:

$$K(u) = \xi \cdot \mu\left(\|\nabla u\|\right) + \alpha \cdot ord\,(u), \tag{3}$$

where $\xi \in (2,3)$, $\alpha \in (0,1)$, μ represents the average operator, *ord* (u) returns the order of u in the evolving sequence. The proposed diffusivity function ψ_u is properly selected, satisfying the conditions required by an edge-stopping function (Weickert, 1998, Barbu, 2014). Thus, it is always positive and monotonically decreasing:

$$\frac{\lambda}{\left(\frac{s_1}{K(u)}\right)^2 + K(u)\log_{10}\left(\frac{s_1}{K(u)}\right)} \geq \frac{\lambda}{\left(\frac{s_2}{K(u)}\right)^2 + K(u)\log_{10}\left(\frac{s_2}{K(u)}\right)}, \forall s_1 \leq s_2 ,$$

implies $\psi_u(s_1) \geq \psi_u(s_2)$. Also, we have $\lim_{s\to\infty} \psi_u(s) = 0$.

Then one can prove the existence and uniqueness of a weak solution in some certain cases. The proposed PDE model accepts solutions if function $s \cdot \psi_u(s)$ is monotonically increasing. Therefore, its derivative must be positive: $\psi_u(s) + s\frac{\partial \psi_u(s)}{\partial s} \geq 0$. We have

$$\psi_1^u(s) + s\frac{\partial \psi_1^u(s)}{\partial s} = \frac{\lambda}{\left(\frac{s}{K(u)}\right)^2 + k(u)\frac{\left|\ln\left(\frac{s}{K(u)}\right)\right|}{\ln(10)}} + s\frac{-\lambda\left(\frac{2s}{K(u)^2} + \frac{K(u)\ln\left(\frac{s}{K(u)}\right)}{s\left|\ln\left(\frac{s}{K(u)}\right)\right|\ln(10)}\right)}{\left(\left(\frac{s}{K(u)}\right)^2 + \frac{K(u)\left|\ln\left(\frac{s}{K(u)}\right)\right|}{\ln(10)}\right)^2}.$$

If $s \leq K(u)$, then the above relation will lead to the following one:

$$\psi_1^u(s) + s\frac{\partial \psi_1^u(s)}{\partial s} = \frac{\lambda\left(\frac{s^2}{K(u)^2} - \frac{K(u)}{\ln(10)}\ln\left(\frac{s}{K(u)}\right) - \frac{2s^2}{K(u)^2} + \frac{K(u)}{\ln(10)}\right)}{\left(\frac{s^2}{K(u)^2} - \frac{K(u)}{\ln(10)}\ln\left(\frac{s}{K(u)}\right)\right)^2} =$$

$$\frac{\lambda\left(\frac{K(u)}{\ln(10)}\left(1 + \ln\left(\frac{K(u)}{s}\right)\right) - \frac{s^2}{K(u)^2}\right)}{\left(\frac{s^2}{K(u)^2} - \frac{K(u)}{\ln(10)}\ln\left(\frac{s}{K(u)}\right)\right)^2} \geq 0 .$$

for $K(u) \geq \ln(10)$, that is a generally verified condition.

Therefore, the PDE-based model here proposed constitutes a forward parabolic equation that is stable and quite likely to have a solution. Two numerical discretization solutions for this PDE are described in the following section.

3. TWO NUMERICAL APPROXIMATION APPROACHES

We consider two robust numerical discretization approaches for the described PDE model. First, we propose an approximation scheme inspired by the discretization algorithm of the Perona-Malik denoising scheme (Perona, 1987). Thus, the following numerical approximation is performed on the PDE model given by (1):

$$u^{n+1}(x,y) = (1-\nu)u^n(x,y) + \varepsilon \sum_{q \in N_p} \psi_u\left(\left\|\nabla u_{p,q}(n)\right\|\right) \cdot \nabla u_{p,q}(n) + \nu u^0(x,y) \quad (4)$$

where $n \in \{0,..,N\}$ and $N_p = \{(x-1,y),(x+1,y),(x,y-1),(x,y+1)\}$ is the set of pixels representing the 4 - neighborhood of the pixel $p = [x, y]$, $\varepsilon \in (0,1)$. The gradient magnitude in a particular direction at iteration n is calculated as follows:

$$\nabla u_{p,q}(n) = u(q,n) - u(p,n) \quad (5)$$

This iterative algorithm applies process (4) on the evolving image for each n from 0 to N, where N is the number of iterations providing optimal smoothing. The restoration scheme is stable and quite consistent to the model given by (1) – (3).

A more consistent discretization solution is achieved by using the finite difference-based method (Johnson, 2008). So, we have:

$$div\left(\psi_u\left(\|\nabla u\|\right)\nabla u\right) = \psi_u\left(\|\nabla u\|\right)\Delta u + \nabla\left(\psi_u\left(\|\nabla u\|\right)\right) \cdot \nabla u \quad (6)$$

The first part of this sum is approximated by using the discretized Laplacian (Johnson, 2008). The second one is calculated as follows:

$$\nabla\left(\psi_u\left(\|\nabla u\|\right)\right) \cdot \nabla u = \left(\frac{\partial}{\partial x}\psi_u\left(\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\right), \frac{\partial}{\partial y}\psi_u\left(\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\right)\right)\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \quad (7)$$

which leads to

$$\nabla\left(\psi_u\left(\|\nabla u\|\right)\right)\cdot\nabla u=\frac{\partial\psi_u}{\partial s}\left(\|\nabla u\|\right)\frac{\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2}} \quad (8)$$

By performing more approximations, from (8) we get:

$$\nabla\left(\psi_u\left(\|\nabla u\|\right)\right)\cdot\nabla u\approx\frac{\partial\psi_u}{\partial s}\left(\sqrt{\left(u_x\right)^2+\left(u_y\right)^2}\right)u_{xy}\left(u_x+u_y\right) \quad (9)$$

where u_x, u_y, u_{xy} can be approximated by using finite differences (Johnson, 2008). So, we obtain the following discretization of the PDE model:

$$u^{n+1}(i,j)=u^n(i,j)+D_1^n(i,j)+D_2^n(i,j)-\lambda\left(u^n(i,j)-u^0(i,j)\right) \quad (10)$$

where $i\in\{0,...,I\}, j\in\{0,...,J\}, n\in\{0,...,N\}$ and D_i^n represent the discretizations of these two components:

$$D_1^n(i,j)=\psi_u\left(\left\|\nabla u^n(i,j)\right\|\right)\left(u^n(i+1,j)+u^n(i-1,j)+u^n(i,j+1)+u^n(i,j-1)-4u^n(i,j)\right) \quad (11)$$

and

$$D_2^n(i,j)=\frac{\partial\psi_u}{\partial s}\left(\sqrt{\left(\frac{\partial u^n(i,j)}{\partial x}\right)^2+\left(\frac{\partial u^n(i,j)}{\partial y}\right)^2}\right)\frac{\partial^2 u^n(i,j)}{\partial x\partial y}\left(\frac{\partial u^n(i,j)}{\partial x}+\frac{\partial u^n(i,j)}{\partial y}\right) \quad (12)$$

where

$$\begin{cases}\dfrac{\partial u^n(i,j)}{\partial x}+\dfrac{\partial u^n(i,j)}{\partial y}=\dfrac{u^n(i+1,j)-u^n(i-1,j)+u^n(i,j+1)-u^n(i,j-1)}{2}\\[2ex]\dfrac{\partial^2 u^n(i,j)}{\partial x\partial y}=\dfrac{u^n(i+1,j+1)-u^n(i+1,j-1)-u^n(i-1,j+1)+u^n(i-1,j-1)}{4}\end{cases} \quad (13)$$

This iterative denoising scheme starts with an initial $[I\times J]$ image and applies repeatedly (13) on u^n, for each n from 0 to N. The developed numerical scheme converges fast, in a low number of steps N, to the solution of the considered PDE model, which is the denoised image u^{N+1}. Both iterative schemes, (4) and (10), can be applied successively on the noisy image to achieve an effective hybrid restoration.

4. EXPERIMENTS AND METHOD COMPARISON

We performed a lot of image restoration experiments using this described nonlinear diffusion-based approach. Our denoising solution was tested on numerous images from the three volumes of the USC - SIPI image database, which were corrupted with various amounts of Gaussian noise. It was observed that it reduces considerably the noise and blurring effect, while preserving important features, such as the edges. Also, it avoids sufficiently the staircasing effect (Buades, 2006). The following PDE model parameters provide optimal results:

$$\lambda = 1.4,\ \varepsilon = 0.3,\ \xi = 2.3,\ \alpha = 0.03,\ \nu = 0.05,\ N = 14. \tag{14}$$

The performance of our restoration scheme was assessed by applying various measures, such as PSNR (Peak Signal to Noise Ratio), Norm of the Error (NE) and Structural Similarity Image Metric (SSIM). Our diffusion-based technique provides better PSNR, NE and SSIM values than both the nonlinear PDE-based methods (Guichard, 2001), such as the two Perona-Malik models and TV Denoising, and the conventional two-dimensional [3 × 3] filters, like the 2D Average, Gaussian, Median or Wiener (Jain, 1989).

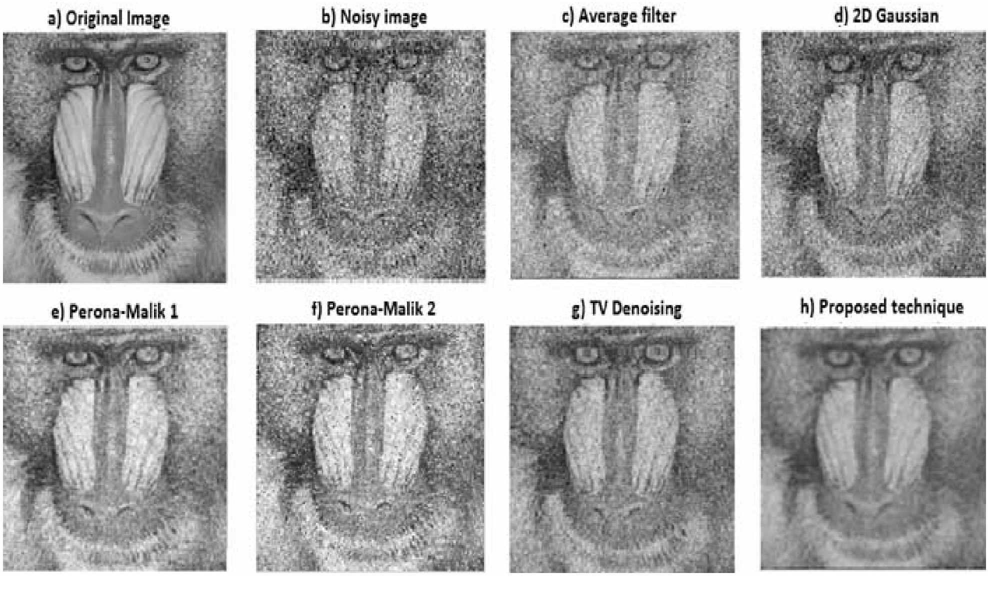

Fig. 1. Method Comparison.

Table 1
SSIM values provided by several methods

Model	This model	Average	Gaussian 2D	Perona-Malik 1	Perona-Malik 2	TV Denoising
SSIM	0.6391	0.5091	0.5412	0.6018	0.6003	0.5918

Some method comparison results are registered in Table 1 and illustrated in Fig. 1. As shown in Table 1, the proposed model gets higher SSIM values than the 2D conventional filters and second-order PDE-based algorithms.

The image restoration results achieved by various image filtering approaches are displayed in Figure 1 (a-h). Thus, the original [512×512] *Baboon* image is displayed in (a), the image affected by a Gaussian noise characterized by $\mu = 0.21$ and $var = 0.02$ is displayed in (b), the denoising achieved by the [3×3] 2D filters (Average and classic Gaussian) in (c) and (d), the results obtained by Perona-Malik schemes in (e) and (f), the restoration of TV Denoising in (g), and the image denoised by our described PDE approach is displayed in (h). Obviously, the last image, corresponding to our technique, looks better than the other image restoration results.

Also, we compared this denoising technique with some of our previous works in this PDE-based domain. Thus, the method described here is based on a more effective edge-stopping function and conductance parameter than the nonlinear second-order anisotropic diffusion model provided in literature (Barbu, 2014). Also, we consider and combine two numerical approximation schemes, which produce a better denoising. The fourth-order PDE-based filtering models proposed by us, such as the one provided in (Barbu, 2015), are more effective for staircase effect removal, while this second-order diffusion technique provides better deblurring and despeckling.

5. CONCLUSIONS

We provided a second-order PDE-based denoising technique based on nonlinear anisotropic diffusion. This approach achieves an effective detail-preserving image noise removal and also overcomes some unintended effects, such as image blurring or staircasing.

The proposed edge-stopping function and conductance parameter represent important contributions of this article. The mathematical investigation of the edge-stopping function selection and the well-posedness of this PDE model, as well as the two numerical approximation schemes proposed in this paper, also represent important contributions of this work.

The performed experiments and the method comparison results prove the effectiveness of the developed method, which outperforms numerous PDE models and conventional 2D image filters. Our future research in the PDE-based image processing field will focus on developing new robust restoration schemes based on higher order PDEs, such as the fourth-order PDE models.

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Authors' contributions*:* TB designed the PDE model and performed the mathematical investigation; TB and AC developed the numerical approximation schemes. TB, AC and CN analyzed data, conducted the experiments and wrote the paper.

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