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MATHEMATICS

HYPERSURFACES OF PROJECTIVE β-CHANGES

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In 1984 C. Shibata investigated the theory of a change which is called a -change of Finsler metric [10]. On the other hand, in 1985 a systematic study of geometry of hypersurfaces in Finsler spaces was given by M. Matsumoto [6]. In § 1, according to Shibata [10] we shall prepare the terminology and notations for the sake of argument. In § 2, we derive a condition for a -change to be projective [10]. In § 3, we find that a totally geodesic hypersurface $F^{n\ 1}$ remains to be a totally geodesic hypersurface $\bar{F}^{n\ 1}$ under the projective -change. In the last section, we get the main result that Finslerian hypersurfaces given by the projective -change are projectively flat on condition the original Finsler space is projectively flat.

Key words: Fundamental tensor, projective change and β-change, hypersurface, projective flat.

1. PRELIMINARIES

Let M^n be an n-dimensional differentiable manifold and $F^n = (M^n, L)$ be a Finsler space equipped with a fundamental function $L(x, y)(y^i = \dot{x}^i)$ on M^n. For a differential one-form $\beta(x, dx) = b_i(x)dx^i$ on M^n, we shall deal with a change of Finsler metric which is defined by $L(x, y) \to \bar{L}(x, y) = f(L(x, y), \beta(x, y))$, where $f(L, \beta)$ is a positively homogeneous function of L and β of degree one. This is called a β-change of the metric.

In this section we shall use the following notations [10]:

$f_1(= f_L) := \partial f / \partial L$, $\quad f_2(= f_\beta) := \partial f / \partial \beta$,

$f_{11}(= f_{LL}) := \partial^2 f / \partial L \partial L$, $\quad f_{12}(= f_{L\beta}) := \partial^2 f / \partial L \partial \beta$,

etcetera.

Since $\overline{L} = f$ is a positively homogeneous function of L and β of degree one, we have

$$f = f_1 L + f_2 \beta\,,\ \ L f_{12} + \beta f_{22} = 0\,,\ L f_{11} + \beta f_{12} = 0\,. \tag{0.1}$$

For the later use we put

$$p = f f_1 / L\,,\ \ q = f f_2\,,\ \ q_0 = f f_{22} \tag{0.2}$$

Paying attention to $l_i = \dot{\partial}_i L$, from (1.1) we have

$$\overline{l}_i = f_1 l_i \quad f_2 b_i \tag{0.3}$$

Differentiating this by y^j, we have the angular metric tensor $\overline{h}_{ij} = \overline{L}\dot{\partial}_i \dot{\partial}_j \overline{L}$ on $\overline{F}^n$:

$$\overline{h}_{ij} = p h_{ij} \quad q_0 m_i m_j \tag{0.4}$$

where the covariant vector m_i is defined by

$$m_i = b_i \qquad y_i / L^2 \tag{1.4$_1$}$$

It is noted that m_i is a non-zero vector orthogonal to y^i. In fact $m_i = 0$ gives $L^2 b_i - \beta y_i = 0$. We differentiate this by y^j and get $\beta g_{ij} - 2 L l_j b_i + b_j y_i = 0$, which leads to a contradiction $g_{ij} - l_i l_j = 0$.

Now, from (1.1), (1.3), (1.4) and (1.4) 1) the fundamental tensor $\overline{g}_{ij} = \dot{\partial}_i \dot{\partial}_j \left(\overline{L}^2 / 2 \right)$ of $\overline{F}^n$ is given by

$$\overline{g}_{ij} = p g_{ij} \quad p_0 b_i b_j \quad p_{\ 1} (b_i y_j \quad b_j y_i) \quad p_{\ 2} y_i y_j\,, \tag{0.5}$$

where we put

$$\begin{aligned} & p_0 = q_0 \quad f_2^{\,2}, \\ & q_{\ 1} = f f_{12} / L, \qquad\qquad p_{\ 1} = q_{\ 1} \quad p f_2 / f\,, \\ & q_{\ 2} = f \ \ f_{11} \quad f_1 / L \ / L^2\,,\ \ p_{\ 2} = q_{\ 2} \quad p^2 / f^2\,. \end{aligned} \tag{1.5$_1$}$$

The reciprocal tensor $\overline{g}^{ij}$ of $\overline{g}_{ij}$ can be written as

$$\overline{g}^{ij} = (1/p) g^{ij} \quad s_0 b^i b^j \quad s_{\ 1} (b^i y^j \quad b^j y^i) \quad s_{\ 2} y^i y^j\,, \tag{0.6}$$

where we put

$$b^i = g^{ij} b_j, \qquad b^2 = g^{ij} b_i b_j, \qquad = b^2 \quad {}^2 / L^2,$$
$$s_0 = \overline{L}^2 q_0 / \; pL^2, \; s_{\ 1} = p_{\ 1} \overline{L}^2 / p \;\; L^2, \tag{1.6$_1$}$$
$$s_{\ 2} = p_{\ 1} \quad pL^2 \quad b^2 \overline{L}^2 \; / \; p \;\; L^2, \qquad = \overline{L}^2 \, p \qquad q_0 / L^2.$$

From the homogeneity it follows that these quantities satisfy

$$\begin{aligned} &q_0 \quad q_{\ 1} L^2 = 0, \qquad q_{\ 1} \quad q_{\ 2} L^2 = \quad p, \\ &p_0 \quad p_{\ 1} L^2 = q, \qquad q \quad pL^2 = f^2, \\ &p_{\ 1} \quad p_{\ 2} L^2 = 0, \qquad s_0 \quad s_{\ 1} L^2 = q / \quad , \\ &s_{\ 1} b^2 \quad s_{\ 2} \quad = p_{\ 1} \; / \; . \end{aligned} \tag{0.7}$$

We denote by the symbol $(_|)$ the h-covariant differentiation with respect to the Cartan connection $C\Gamma = \left(F^i_{j\,k}, N^i{}_j, C^i_{j\,k}\right)$ and put

$$2E_{jk} = b_{j|k} \quad b_{k|j}, \tag{0.8}$$

(1.8) $\quad 2E_{jk} = b_{j|k} + b_{k|j}, \; 2F_{jk} = b_{j|k} - b_{k|j}.$

Now we deal with well-known functions $G^i(x,y)$ which are $(2)p$-homogeneous in y^i and are written as $G^i = \gamma_j{}^i{}_k y^j y^k / 2$, by putting $\gamma_j{}^i{}_k = g^{ir}\left(\partial_k g_{jr} + \partial_j g_{kr} - \partial_r g_{jk}\right)/2$.

Owing to (1.5) and (1.6), a straightforward calculation leads to

(1.9) $\quad \overline{G}^i(x,y) := \left(\overline{\gamma}_j{}^i{}_k y^j y^k\right)/2 = G^i + D^i,$

where the vector D^i is given by

(1.9) 1) $\; D^i = (q/p) F^i{}_0 + \left(pE_{00} - 2qF_{r0} b^r\right)\left(s_{-1} y^i + s_0 b^i\right)/2,$

$F^i{}_j = g^{ir} F_{rj}$, and the subscript 0 (excluding s_0) means the contraction by y^i.

2. RELATION BETWEEN PROJECTIVE CHANGE AND -CHANGE

For two Finsler spaces $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$, if any geodesic on F^n is also a geodesic on $\overline{F}^n$ and the inverse is true, the change $L \to \overline{L}$ of the metric is called *projective*. A geodesic on F^n is given by a system of differential equations

$$dy^i / dt \quad 2G^i(x,y) = \quad y^i, \quad y^i = dx^i / dt, \tag{1.1}$$

where $\tau = \left(d^2s/dt^2\right)/\left(ds/dt\right), G^i(x,y)$ are (2) p -homogeneous functions in y^i. We are now in a position to find a condition for a β -change to be projective. For this purpose we deal with Euler-Lagrange equations $B_i = 0$, where B_i is defined by $B_i = \partial_i L - d(\dot{\partial}_i L)/dt$. From (1.1) and (1.2), we have $Ldf_1/dt + \beta df_2/dt = 0$, so that the Euler-Lagrange differential equations $\overline{B}_i = 0$ for $\overline{F}^n$ are given by $\overline{B}_i = f_1 B_i + 2 f_2 F_{0i} - m_i df_2/dt = 0$.

In virtue of (1.2) and (1.4) 1), $\overline{B}_i$ are written as

$$f\overline{B}_i = pLB_i \quad q_0 LB_r m^r m_i \quad A_i, \tag{1.2}$$

where the covariant vector A_i is defined by

$$A_i = 2qF_{0i} \quad q_0 E_{00} m_i. \tag{1.3}$$

From (2.2) we get

Proposition 2.1[10]. *A* β*-change* $L \to \overline{L} = f(L,\beta)$ *of the metric is projective if and only if the covariant vector* A_i *in* (2.3) *vanishes identically.*

Proof. Since $B_i = 0$ (resp. $\overline{B}_i = 0$) are equations of a geodesic on F^n (resp. $\overline{F}^n$), we immediately obtain $A_i = 0$ if a β-change is projective. Conversely if $A_i = 0$ holds, then (2.2) shows that $B_i = 0$ lead to $\overline{B}_i = 0$. On the other hand, we observe from (2.2) and $A_i = 0$ that $\overline{B}_i = 0$ give $pLB_i + q_0 LB_r m^r m_i = 0$. Contracting this by m^i and referring to $m^2 = \nu$, $pL + \nu q_0 L \neq 0$, we get $B_r m^r = 0$, so that $B_i = 0$ hold. Consequently any geodesic remains to be a geodesic by a -change.

3. HYPERSURFACES GIVEN BY A PROJECTIVE -CHANGE

Hereafter, we assume that metrics L^2 and $\overline{L}^2$ are positive-definite respectively and we consider hypersurfaces. For their theory we refer to [2]-[4], [6], [9]-[13]. According to [6], a hypersurface M^{n-1} of the underlying smooth manifold M^n may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where u^α are Gaussian coordinates on M^{n-1} and Greek indices run from 1 to

$n-1$. Here, we shall assume that the matrix consisting of the projection factors $B^{\ i}_{\alpha} = \partial x^i / \partial u^\alpha$ is of rank $n-1$. The following notations are also employed : $B_{\alpha\ \beta}^{\ i} := \partial^2 x^i / \partial u^\alpha \partial u^\beta$, $B_{\bar{0}\ \beta}^{\ i} := v^\alpha B_{\alpha\ \beta}^{\ i}$, $B^{ij\cdots}_{\quad\ \alpha\beta\cdots} := B_\alpha^{\ i} B_\beta^{\ j} \cdots$. If the supporting element y^i at a point (u^α) of M^{n-1} is assumed to be tangential to M^{n-1}, then we may write $y^i = B^{\ i}_{\alpha}(u) v^\alpha$, so that v^α is thought of as the supporting element of M^{n-1} at the point (u^α). Since the function $\underline{L}(u,v) := L(x(u), y(u,v))$ gives rise to a Finsler metric of M^{n-1}, we get an $(n-1)$-dimensional Finsler space $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$.

At each point (u^α) of F^{n-1}, the unit normal vector $N^i(u,v)$ is defined by

$$g_{ij} B^{\ i} N^j = 0, \quad g_{ij} N^i N^j = 1. \tag{2.1}$$

If $(B^{\alpha}{}_{i}, N_i)$ is the inverse matrix of $(B^{\ i}_{\alpha}, N^i)$, we have

$$B^{\ i} B_{\ \ i} = \quad , \quad B^{\ i} N_i = 0, \quad N^i B_{\ \ i} = 0, \quad N^i N_i = 1, \tag{2.2}$$

and further

$$B^{\ i} B_{\ \ j} \quad N^i N_j = \ ^{i}_{\ j}. \tag{2.3}$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get $B^{\alpha}{}_{i} = g^{\alpha\beta} g_{ij} B^{\ j}_{\beta}$, $N_i = g_{ij} N^j$.

For the induced Cartan connection $IC\Gamma = (F^{\alpha}_{\beta\gamma}, N^{\alpha}{}_{\beta}, C^{\alpha}_{\beta\gamma})$ on F^{n-1}, the normal curvature vector H_α is given by

$$H \ : \ N_i (B_{\bar{0}}^{\ \ i} \quad N^i{}_j B^{\ j}), \tag{2.4}$$

where $B_{\bar{0}\alpha}^{\ \ i} = B_{\beta\alpha}^{\ \ i} v^\beta$.

Let's introduce in $\overline{F}^n = (M^n, \overline{L})$ the Cartan connection $C\overline{\Gamma}$. We now consider a Finslerian hypersurface $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$ of F^n and another Finslerian hypersurface $\overline{F}^{n-1} = (M^{n-1}, \underline{\overline{L}}(u,v))$ of the $\overline{F}^n$ given by the β-change. Let N^i be unit normal vector at each point of F^{n-1}, and $(B^{\alpha}{}_{i}, N_i)$ be the inverse matrix of $(B^{\ i}_{\alpha}, N^i)$. The functions $B^{\ i}_{\alpha}(u)$ may be considered as components of $n-1$ linearly independent vectors tangent to F^{n-1} and they are invariant under the β-change. And so we shall show that unit normal vector $\overline{N}^i(u,v)$ of $\overline{F}^{n-1}$ is uniquely determined by

$$\bar{g}_{ij} B^{\ i} \bar{N}^j = 0, \quad \bar{g}_{ij} \bar{N}^i \bar{N}^j = 1. \tag{2.5}$$

The fundamental tensor $\bar{g}_{ij} = \left(\partial^2 \bar{L}^2 / \partial y^i \partial y^j\right)/2$ of the Finsler space $\bar{F}^n$ given by the β-change is as follows:

$$\bar{g}_{ij} = pg_{ij} \quad p_0 b_i b_j \quad p_{\ 1}(b_i y_j \quad b_j y_i) \quad p_{\ 2} y_i y_j,$$

Now contracting (3.1) by v^{α}, we immediately get

$$y_i N^i = 0. \tag{2.6}$$

Further contracting (1.5) by $N^i N^j$ and paying attention to (3.1), (3.5) and (3.6), we have

$$\bar{g}_{ij} N^i N^j = p \quad p_0 (b_i N^i)(b_j N^j). \tag{2.7}$$

Then we obtain

$$\bar{g}_{ij} \quad N^i \Big/ \sqrt{p \quad p_0 (b_k N^k)^2} \quad N^j \Big/ \sqrt{p \quad p_0 \ b_k N^k \ ^2} \ = 1, \tag{2.8}$$

provided $p + p_0 \left(b_k N^k\right)^2 > 0$.

Therefore we can put

$$\bar{N}^i = N^i \Big/ \sqrt{p \quad p_0 \ b_k N^k \ ^2}, \tag{2.9}$$

where we have chosen the sign "+" in order to fix an orientation.

On using (1.5),(3.1),(3.6) and (3.9), the first condition of (3.5) gives us

$$b_i N^i \quad p_0 b_j B^{\ j} \quad p_{\ 1} y_j B^{\ j} \ = 0. \tag{2.10}$$

Now, assuming that $p_0 b_j B_{\alpha}^{\ j} + p_{-1} y_j B_{\alpha}^{\ j} = 0$ and contracting this by v^{α}, we find $p_0 \beta + p_{-1} L^2 = 0$. By (1.7) this equation leads us to $ff_2 = 0$. Thus we have $f_2 = 0$ because of $f \neq 0$. This fact means $\bar{L} = f(L)$ and contradicts the definition of a β-change of the metric. So, (3.10) gives us

$$b_i N^i = 0. \tag{2.11}$$

Consequently (3.9) is rewritten as

$$\bar{N}^i = N^i \Big/ \sqrt{p} \qquad (p > 0), \tag{2.12}$$

and then it is clear $\overline{N}^i$ satisfies (3.5). Summarizing the above, we obtain

Proposition 3.1 [3]. *For a field of linear frame* $(B_1^i, \cdots, B_{n-1}^i, N^i)$ *of* F^n*, there exists a field of linear frame* $(B_1^i, \cdots, B_{n-1}^i, \overline{N}^i)$ *of the* $\overline{F}^n$ *given by the* β*-change such that* (3.5) *is satisfied along* $\overline{F}^{n-1}$*, and then we get* (3.11).

The quantities $\overline{B}^\alpha{}_i$ are uniquely defined along $\overline{F}^{n-1}$ by $\overline{B}^\alpha{}_i = \overline{g}^{\alpha\beta}\overline{g}_{ij}B_\beta{}^j$, where $(\overline{g}^{\alpha\beta})$ is the inverse matrix of $(\overline{g}_{\alpha\beta})$.

Let $(\overline{B}^\alpha{}_i, \overline{N}_i)$ be the inverse matrix of $(B_\alpha{}^i, \overline{N}^i)$, and then we have

$$B\,{}^i\overline{B}_i = \quad ,\ B\,{}^i\overline{N}_i = 0,\ \overline{N}^i\overline{B}_i = 0,\ \overline{N}^i\overline{N}_i = 1, \tag{2.13}$$

and further

$$B\,{}^i\overline{B}_j \quad \overline{N}^i\overline{N}_j = \quad {}^i{}_j. \tag{2.14}$$

We also get $\overline{N}_i = \overline{g}_{ij}\overline{N}^j$, that is,

$$\overline{N}_i = \sqrt{p}N_i. \tag{2.15}$$

We now assume that a β-change of the metric is projective.

Using (1.9), (1.9)1) and Proposition 2.1, we have

$$D^i := \overline{G}^i \quad G^i = \quad y^i, \qquad = qE_{00}/2\overline{L}^2. \tag{2.16}$$

Since $D_j^i := \dot{\partial}_j D^i$ and $\dot{\partial}_j G^i = N^i{}_j$, the above gives

$$D^i{}_j = \overline{N}^i_j \quad N^i{}_j = y^i\,\dot{}\ {}_j \qquad {}^i{}_j. \tag{2.17}$$

Further contracting (3.17) by $N_i B_\alpha^j$, we get

$$N_i D^i_j B^j = 0. \tag{2.18}$$

If each geodesic of F^{n-1} with respect to the induced metric is also a geodesic of F^n, then F^{n-1} is called totally geodesic. A totally geodesic hypersurface F^{n-1} is characterized by $H_\alpha = 0$.

From (3.4), (3.15) and (3.17) we have

$$\overline{H} = \sqrt{p}\ \ H \quad N_i D^i_j B^j\ . \tag{2.19}$$

Thus from (3.18) we obtain $\overline{H}_\alpha = \sqrt{p}H_\alpha$. Hence we have

Theorem 3.1. *A hypersurface* F^{n-1} *of a Finsler space* $F^n (n > 3)$ *is totally geodesic, if and only if the hypersurface* $\overline{F}^{n-1}$ *of the space* $\overline{F}^n$*, obtained from* F^n *by a projective* β*-change, is totally geodesic.*

4. HYPERSURFACES OF PROJECTIVELY FLAT FINSLER SPACES

In this section, we shall consider a projective β-change and we are concerned with the Berwald connection $B\Gamma$ on $F^n = (M^n, L)$ and $B\overline{\Gamma}$ on $\overline{F}^n = (M^n, \overline{L})$. A projective change $L \to \overline{L}$ is defined as follows: If any geodesic on $F^n = (M^n, L)$ is also a geodesic on $\overline{F}^n = (M^n, \overline{L})$ and the inverse is true, the change $L \to \overline{L}$ of the metric is called projective. For details see [5],[7],[8].

In the theory of projective changes in Finsler spaces, we have two essential projective invariants, one is the Weyl torsion tensor $W^h_{\ ij}$ and the other is the Douglas tensor $D_i{}^h{}_{jk}$, so that under the projective β-change, we get $\overline{W}^h_{\ ij} = W^h_{\ ij}$ and $\overline{D}_i{}^h{}_{jk} = D_i{}^h{}_{jk}$.

Now we are concerned with a projectively flat Finsler space defined as follows: If there exists a projective change $L \to \overline{L}$ of a Finsler space $F^n = (M^n, L)$ such that the Finsler space $\overline{F}^n = (M^n, \overline{L})$ is a locally Minkowski space, F^n is called projectively flat. We have already known the following.

Theorem A [4]. *A Finsler space* $F^n (n > 2)$ *is projectively flat, if and only if* $W^h_{\ ij} = 0$ *and* $D_i{}^h{}_{jk} = 0$.

Theorem B [11]. *If the Finsler space* $F^n (n > 3)$ *is projectively flat, then the totally geodesic hypersurface* F^{n-1} *is also projectively flat.*

Thus from Theorem 3.1, Theorem A and Theorem B, we have

Theorem 4.1. *Let* $F^n (n > 3)$ *be a projectively flat Finsler space. If the hypersurface* F^{n-1} *is totally geodesic, then the hypersurface* $\overline{F}^{n-1}$ *of the space* $\overline{F}^n$*, obtained from* F^n *by a projective -change, is projectively flat.*

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