B.-Y. CHEN INEQUALITIES FOR SLANT SUBMANIFOLDS IN S-SPACE-FORMS

SIMONA COSTACHE

In this article, we investigate sharp inequalities involving Chen invariants for a slant submanifold M of a S-space-form $\tilde{M}(c)$, tangent to the structure vector fields of the ambient space.

Key words: S-space-form, slant submanifold, Chen invariants.

1. PRELIMINARIES

Let $(\tilde{M}, g)$ be a Riemannian manifold. Then $\tilde{M}$ is said to be a metric f-manifold if there exist a (1,1) tensor field f, s global unit vector fields $\xi_1, ..., \xi_s$ (called structure vector fields) and s 1-forms $\eta_1, ..., \eta_s$ on $\tilde{M}$ which satisfy:

$$f^2X = -X + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha, \; f\xi_\alpha = 0, \; \eta_\alpha \circ f = 0,$$

$$g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \; g(X, \xi_\alpha) = \eta_\alpha(X),$$

for any vector fields X, Y on $\tilde{M}$ and $\alpha = 1,..., s$.

The f-structure f is said to be normal if

$$[f, f] + 2\sum_\alpha \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f,f]$ is the Nijenhuis torsion of f.

Let F denote the fundamental 2-form in $\tilde{M}$ given by

$$F(X,Y) = g(X, fY),$$

for any $X, Y \in T\tilde{M}$.

$\tilde{M}$ is called an S-manifold if the structure is normal and $F = d\eta_\alpha$, for any $\alpha = 1,..., s$.

A plane section π in $T_p\tilde{M}$ is called an f-section if it is spanned by X and fX, where X is a unit tangent vector field orthogonal to the distribution **M** spanned by the structure vector fields. The sectional curvature $K(\pi)$ of an

f- section π is called f- sectional curvature. An S -manifold is called an S- space-form if it has constant f- sectional curvature c and it is denoted by $\tilde{M}(c)$. Then its curvature tensor $\tilde{R}$ is expressed by

$$\tilde{R}(X,Y,Z,W) = \sum_{\alpha,\ \beta=1}^{s}[g(fX,fZ)\eta_\alpha(Y)\eta_\beta(W) - g(fX,fW)\eta_\alpha(Y)\eta_\beta(Z) +$$
$$+g(fY,fW)\eta_\alpha(X)\eta_\beta(Z) - g(fY,fZ)\eta_\alpha(X)\eta_\beta(W)] +$$
$$+\frac{c+3s}{4}[g(fX,\ fZ)g(fY,fW) - g(fX,\ fW)g(fY,fZ)] +$$
$$+\frac{c-s}{4}[g(X,fZ)g(Y,fW) - g(X,fW)g(Y,fZ) + 2g(X,fY)g(Z,fW)],$$

for any $X,Y,Z,W \in T\tilde{M}$.

Let M be a submanifold isometrically immersed in $\tilde{M}$. We denote by TM and $T^{\perp}M$ the tangent and the normal bundles of M respectively.

For any $X \in TM$, we write $fX = TX + NX$, where TX (respectively NX) denotes the tangential (respectively normal) component of fX.

The equation of Gauss is given by

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,Z),\ h(Y,W)) + g(h(X,W),\ h(Y,Z)),$$

for any vectors X,Y,Z,W tangent to M.

The mean curvature vector H is defined by $H = \frac{1}{\dim M} trace h$.

From now on, let $n+s$ (respectively m) be the dimension of M (respectively $\tilde{M}$).

The scalar curvature of M at $p \in M$ is defined by

$$\tau(p) = \sum_{1 \le i < j \le n+s} K(e_i \wedge e_j),$$

where $p \in M$ and $\{e_1,\dots,\ e_n,\ \xi_1,\dots,\ \xi_s\}$ is an orthonormal basis of T_pM and $K(X \wedge Y)$ denotes the sectional curvature of M associated with the plane section spanned by $X,Y \in TM$.

We denote by

$$h_{ij}^r = g(h(e_i,\ e_j),\ e_r),\ i,\ j \in \{1,\dots,n+s\},\ r \in \{n+s+1,\dots,m\};$$

then we have

$$\|H\|^2 = \frac{1}{(n+s)^2}\sum_{r=n+s+1}^{m}\Big(\sum_{i=1}^{n+s} h_{ii}^r\Big)^2,\ \|h\|^2 = \sum_{r=n+s+1}^{m}\sum_{i,\ j=1}^{n+s}(h_{ij}^r)^2.$$

Also we put

$$\|T\|^2 = \sum_{i,\ j=1}^{n+s} g^2(Te_i, e_j),$$

where $\{e_1,...,\ e_n,\ \xi_1,...,\ \xi_s\}$ is an orthonormal basis of T_pM and $\{e_{n+s+1},...e_m\}$ is an orthonormal basis of $T_p^?M$.

If the structure vector fields are tangent to M, we denote by $\mathbf{L}$ the orthogonal distribution to $\mathbf{M}$ in TM and we can consider the orthogonal direct decomposition $TM = \mathbf{L} \oplus \mathbf{M}$.

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1,...,\ e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of the r-plane section L is given by:

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq r} K(e_\alpha \wedge e_\beta)$$

and we denote by

$$\Psi(L) = \sum_{1 \leq i < j \leq r} g^2(Te_i,\ e_j).$$

For an integer $k \geq 0$, we denote by $S(n,k)$ the finite set consisting of k-tuples $(n_1,...,n_k)$ of integers ≥ 2 satisfying $n_1 < n,\ n_1 + ... + n_k \leq n$. Denote by $S(n)$ the set of k-tuples with $k \geq 0$ for a fixed n.

For each k-tuples $(n_1,...,\ n_k) \in S(n)$, Chen introduced a Riemannian invariant defined by:

$$\delta(n_1,...,\ n_k)(p) = \tau(p) - S(n_1,...,\ n_k)(p),$$

where $S(n_1,...,\ n_k)(p) = \inf\{\tau(L_1) + ... + \tau(L_k)\}$ and at each point $p \in M$, $L_1,...,\ L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j,\ \ j = 1,...,\ k$.

For each $(n_1,...,\ n_k) \in S(n)$, let:

$$d(n_1,...,\ n_k) = \frac{(n+s)^2\left(n+s+k-1-\sum_{j=1}^{k} n_j\right)}{2\left(n+s+k-\sum_{j=1}^{k} n_j\right)},$$

$$b(n_1,\ldots,\ n_k)=\frac{1}{2}\left[n(n-1)-\sum_{j=1}^{k} n_j(n_j-1)\right].$$

The submanifold M is said to be invariant if N is identically zero, that is, if $fX\in TM$, for any $X\in TM$ and it is said to be anti-invariant if T is identically zero, that is, if $fX\in T^{\perp}M$, for any $X\in TM$. Moreover, if for each nonzero vector $X\in T_pM-\mathbf{M}_p$, we consider the angle $\theta(X)$ between fX and T_pM, then the submanifold is said to be θ-slant if such angle is a constant, which is independent on the choice of $p\in M$ and $X\in T_pM-\mathbf{M}_p$. The angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant submanifolds tangent to the structure vector fields are slant submanifolds with slant angle $\theta=0$ and $\theta=\frac{\pi}{2}$, respectively. A slant immersion which is not invariant nor anti-invariant is called a proper slant immersion.

2. INEQUALITIES

We recall the following lemma due to Chen.

Lemma 1. *Let* $a_1,\ldots a_n,\ b$ *be* $n+1$ $(n\geq 2)$ *real numbers such that:*

$$\left(\sum_{i=1}^{n} a_i\right)^2=(n-1)\left(\sum_{i=1}^{n} a_i^2+b\right).$$

Then, $2a_1a_2\geq b$, *with the equality holding if and only if* $a_1+a_2=a_3=\ldots=a_n$.

Lemma 2. *Let M be an* $(n+s)$*-dimensional submanifold, tangent to the structure vector fields* $\xi_1,\ldots,\ \xi_s$ *of a* m*-dimensional S-space form* $\tilde{M}(c)$. *Let* $n_1,\ldots,\ n_k$ *be integers* ≥ 2 *satisfying* $n_1<n,\ n_1+\ldots+n_k\leq n$. *For* $p\in M$, *let* $L_j\subset T_pM$ *be subspaces of* T_pM, *orthogonal to the structure vector fields* $\xi_1,\ldots,\ \xi_s$ *such that* $\dim L_j=n_j, \forall j\in\{1,\ldots,\ k\}$. *Then, we have:*

$$\tau(p)-\sum_{j=1}^{k}\tau(L_j)\leq d(n_1,\ldots,n_k)\|H\|^2+b(n_1,\ldots,n_k)\frac{c+3s}{4}+$$

$$+\frac{3(c-s)}{8}\left[\|T\|^2-2\sum_{j=1}^{k}\Psi(L_j)\right]+ns.$$

Proof. Let $p\in M$ and $\{e_1,..., e_n, e_{n+1}=\xi_1,..., e_{n+s}=\xi_s\}$ be an orthonormal basis of T_pM.

From the Gauss equation we get

$$2\tau=(n+s)^2\|H\|^2-\|h\|^2+\frac{c+3s}{4}n(n-1)+3\frac{c-s}{4}\|T\|^2+2ns.$$

Denoting by

$$\eta=2\tau-2d(n_1,...,n_k)\|H\|^2-\frac{c+3s}{4}n(n-1)-3\frac{c-s}{4}\|T\|^2-2ns,$$

it follows that

$$(n+s)^2\|H\|^2=\left(\eta+\|h\|^2\right)\gamma, \tag{1}$$

where

$$\gamma=n+s+k-\sum_{j=1}^{k}n_j.$$

Let e_{n+s+1} be a unit normal vector at p parallel to $H(p)$ and $\{e_{n+s+1},...,e_m\}$ an orthonormal basis of $T_p^{\perp}M$.

We denote by $a_i=h_{ii}^{n+s+1}=g(h(e_i, e_i), e_{n+s+1})$.

The relation (1) becomes

$$\left(\sum_{i=1}^{n+s}a_i\right)^2=\gamma\{\eta+\sum_{i\neq j}(h_{ij}^{n+s+1})^2+ +\sum_{i=1}^{n+s}(h_{ii}^{n+s+1})^2+\sum_{r=n+s+2i,}^{m}\sum_{j=1}^{n+s}(h_{ij}^{r})^2\}. \tag{2}$$

Let $L_1,..., L_k$ be k mutually orthogonal subspaces of T_pM, $\dim L_j=n_j$, defined by:

$L_1=Sp\left\{e_1,..., e_{n_1}\right\}$,

$L_2=Sp\left\{e_{n_1+1},..., e_{n_1+n_2}\right\}$,

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$L_k=Sp\left\{e_{n_1+...+n_{k-1}+1},..., e_{n_1+...+n_k}\right\}$.

We denote by D_j, $j=1,..., k$ the sets:

$D_1=\{1,\dots,\ n_1\}$,
$D_2=\{n_1+1,\dots,\ n_1+n_2\}$,
`
$D_k=\{n_1+\dots+n_{k-1}+1,\dots,\ n_1+\dots+n_k\}$.

Also we put:
$b_1=a_1$,
$b_2=a_2+\dots+a_{n_1}$,
$b_3=a_{n_1+1}+\dots+a_{n_1+n_2}$,
`
$b_{k+1}=a_{n_1+\dots n_{k-1}+1}+\dots+a_{n_1+\dots+n_k}$,
$b_{k+2}=a_{n_1+\dots+n_k+1}$,
`
$b_{\gamma+1}=a_{n+s}$.

Then the relation (2) is equivalent to

$$\left(\sum_{i=1}^{\gamma+1} b_i\right)^2=\gamma[\eta+\sum_{i=1}^{\gamma+1} b_i^2+\sum_{i\neq j}(h_{ij}^{n+s+1})^2+\sum_{r=n+s+2}^{m}\sum_{i,\ j=1}^{n+s}(h_{ij}^r)^2-$$
$$-2\sum_{2\leq\alpha_1<\beta_1\leq n_1} a_{\alpha_1}a_{\beta_1}-2\sum_{\substack{\alpha_2<\beta_2\\ \alpha_2,\ \beta_2\in D_2}} a_{\alpha_2}a_{\beta_2}-\dots-2\sum_{\substack{\alpha_k<\beta_k\\ \alpha_k,\ \beta_k\in D_k}} a_{\alpha_k}a_{\beta_k}].$$

Applying the algebraic lemma we have:

$$2b_1b_2\geq\eta+\sum_{i\neq j}(h_{ij}^{n+s+1})^2+\sum_{r=n+s+2}^{m}\sum_{i,\ j=1}^{n+s}(h_{ij}^r)^2-$$
$$-2\left(\sum_{2\leq\alpha_1<\beta_1\leq n_1} a_{\alpha_1}a_{\beta_1}+\sum_{\substack{\alpha_2<\beta_2\\ \alpha_2,\ \beta_2\in D_2}} a_{\alpha_2}a_{\beta_2}+\dots+\sum_{\substack{\alpha_k<\beta_k\\ \alpha_k,\ \beta_k\in D_k}} a_{\alpha_k}a_{\beta_k}\right),$$

which is equivalent to:

$$\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1}+\dots.+\sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k}\geq\frac{1}{2}\left[\eta+\sum_{i\neq j}(h_{ij}^{n+s+1})^2+\sum_{r=n+s+2}^{m}\sum_{i,\ j=1}^{n+s}(h_{ij}^r)^2\right],$$

with $\alpha_i,\beta_i\in D_i,\forall i=1,\dots,k.$

From the Gauss equation we obtain:

$$\tau(L_j)=\frac{n_j(n_j-1)(c+3s)}{8}+\frac{3(c-s)}{4}\Psi(L_j)+$$

$$+\sum_{r=n+s+1}^{m}\sum_{\alpha_j<\beta_j}\left[h^r_{\alpha_j\alpha_j}h^r_{\beta_j\beta_j}-\left(h^r_{\alpha_j\beta_j}\right)^2\right].$$

It follows that

$$\sum_{j=1}^{k}\sum_{r=n+s+1}^{m}\sum_{\alpha_j<\beta_j}\left[h^r_{\alpha_j\alpha_j}h^r_{\beta_j\beta_j}-\left(h^r_{\alpha_j\beta_j}\right)^2\right]\geq\frac{\eta}{2}+\frac{1}{2}\sum_{r=n+s+1}^{m}\sum_{(\alpha,\,\beta)\notin D^2}(h^r_{\alpha\beta})^2+$$

$$+\frac{1}{2}\sum_{r=n+s+2}^{m}\sum_{j=1}^{k}\left(\sum_{\alpha_j\in D_j}h^r_{\alpha_j\alpha_j}\right)^2\geq\frac{\eta}{2},$$

where $D^2=(D_1\times D_1)\cup\ldots\cup(D_k\times D_k)$.

Thus

$$\sum_{j=1}^{k}\tau(L_j)\geq\frac{\eta}{2}+\sum_{j=1}^{k}\left[\frac{n_j(n_j-1)(c+3s)}{8}+\frac{3(c-s)}{4}\Psi(L_j)\right]=$$

$$=\tau-d(n_1,\ldots,\;n_k)\|H\|^2-\frac{c+3s}{8}n(n-1)-3\frac{c-s}{8}\|T\|^2-ns+$$

$$+\sum_{j=1}^{k}\left[\frac{n_j(n_j-1)(c+3s)}{8}+\frac{3(c-s)}{4}\Psi(L_j)\right],$$

which is equivalent with the relation that we want to prove.

In particular, for slant submanifolds we derive:

Theorem 3. *Let M be an* $(n+s)$*-dimensional* θ*-slant submanifold, tangent to the structure vector fields of a m -dimensional S-space form* $\tilde{M}(c)$*. Let* $n_1,\ldots,\;n_k$ *be integers* ≥ 2 *satisfying* $n_1<n,\;n_1+\ldots+n_k\leq n$ *. For* $p\in M$*, let* $L_j\subset T_pM$ *be subspaces of* T_pM*, orthogonal to the structure vector fields such that* $\dim L_j=n_j,\forall j\in\{1,\ldots,\;k\}$*. Then, we have:*

$$\tau(p)-\sum_{j=1}^{k}\tau(L_j)\leq d(n_1,\ldots,n_k)\|H\|^2+b(n_1,\ldots,n_k)\frac{c+3s}{4}+$$

$$+\frac{3(c-s)}{8}\left[n\cos^2\theta-2\sum_{j=1}^{k}\Psi(L_j)\right]+ns.$$

The equality case of the inequality holds at $p\in M$ *if and only if there exists an orthonormal basis* $\{e_1,\ldots,\;e_n,\;e_{n+1}=\xi_1,\ldots,\;e_{n+s}=\xi_s\}$ *of* T_pM *and an orthonormal basis* $\{e_{n+s+1},\ldots,\;e_m\}$ *of* $T_p^{?}M$ *such that the shape operators of M in* $\tilde{M}(c)$ *at p have the following forms:*

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix},$$

$$a_1 + \dots + a_{n_1} = a_{n_1+1} + \dots + a_{n_1+n_2} = \dots =$$
$$= a_{n_1+\dots+n_{k-1}+1} + \dots + a_{n_1+\dots+n_k} = a_{n_1+\dots+n_k+1} = \dots = a_{n+s},$$

$$A_r = \begin{pmatrix} A_1^r & 0 & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 & \dots & 0 \\ 0 & 0 & A_k^r & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$A_j^r \in M_{n_j}(\mathbf{R}), {}^t A_j^r = A_j^r,\ TrA_j^r = 0, \forall j = \overline{1,\ k}, \forall r \in \{n+s+2, \dots, m\}.$$

Proof. For a θ-slant submanifold of a S-space form we have $\|T\|^2 = n\cos^2\theta.$

Equality at a point $p \in M$ holds if and only if achieves the equality in all the previous inequalities and we have the equality in the algebraic lemma:

$$h_{\alpha\beta}^r = 0, \forall r = \overline{n+s+1,\ m}, \forall(\alpha,\ \beta) \notin D^2,$$
$$\sum_{\alpha_j \in D_j} h_{\alpha_j\alpha_j}^r = 0, \forall r = \overline{n+s+2,\ m}, \forall j = \overline{1,\ k},$$
$$b_1 + b_2 = b_3 = \dots = b_{\gamma+1}.$$
$$A_r = (h_{ij}^r)_{\substack{i,\ j=\overline{1,\ n+s} \\ r=\overline{n+s+1,\ m}}}.$$

For invariant submanifolds we have the following:

Corollary 4. *Let* M *be an* $(n+s)$*-dimensional invariant submanifold, tangent to the structure vector fields of a* m*-dimensional S-space form* $\tilde{M}(c)$. *Let* $n_1, \dots,\ n_k$ *be integers* ≥ 2 *satisfying* $n_1 < n,\ n_1 + \dots + n_k \leq n.$ *For* $p \in M$, *let* $L_j \subset T_pM$ *be subspaces of* T_pM, *orthogonal to the structure vector fields such that* $\dim L_j = n_j, \forall j \in \{1, \dots,\ k\}$. *Then, we have:*

$$\tau(p) - \sum_{j=1}^{k} \tau(L_j) \leq b(n_1, \dots,\ n_k)\frac{c+3s}{4} + \frac{3(c-s)}{8}\left[n - 2\sum_{j=1}^{k} \Psi(L_j)\right] + ns.$$

Proof. It is known that every invariant submanifold of a S-space form is minimal.

For anti-invariant submanifolds we obtain:

Corollary 5. *Let M be an $(n+s)$-dimensional anti-invariant submanifold, tangent to the structure vector fields of a m-dimensional S-space form $\tilde{M}(c)$. Let $n_1,\ldots, n_k$ be integers ≥ 2 satisfying $n_1 < n,\ n_1 + \ldots + n_k \leq n$. For $p \in M$, let $L_j \subset T_pM$ be subspaces of T_pM, orthogonal to the structure vector fields such that $\dim L_j = n_j, \forall j \in \{1,\ldots,\ k\}$. Then, we have:*

$$\tau(p) - \sum_{j=1}^{k} \tau(L_j) \leq d(n_1,\ldots,n_k)\|H\|^2 + b(n_1,\ldots,n_k)\frac{c+3s}{4} + ns.$$

3. CONCLUSIONS

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Faculty of Mathematics and Computer Science,
University of Bucharest, 14 Academiei Street,
010014 Bucharest, Romania
E-mail: simona_costache2003@yahoo.com.