SPACES WITH VECTOR NORM AND BOUNDED LINEAR OPERATORS

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In this paper we give some results about the operators with values into spaces with vector norm or defined on such spaces. Regular operators between spaces with vector norm are also considered.

The terminology is mainly that of [4] but some definitions are specified in this paper.

Key words: Spaces with vector norm, regular operators.

PRELIMINARIES

We call a ν-*normed vector space*, any vector space F endowed with a vector norm $P : F \to X$ where X is a vector lattice. We denote by F_X such a space and we also denote $\|f\|_X = P(f)$.

A subset A of the space F_X is said to be (ν)-*bounded* if there exists an element $a \in X$ such that $\|f\|_X \le a,\ \forall f \in A.$

We recall also that a sequence $(f_n)_{n\in\mathrm{N}}$ of elements of F_X is said to (ν)-*converge* to an element $f \in F_X$, and we denote $f = (\nu)\text{-}\lim\limits_n f_n$ if

$$(o)\text{-}\lim_n \|f_n - f\|_X = \mathbf{0} \tag{0.1}$$

where (*o*)-*lim* means the limit with respect to the order relation.

A ν-normed vector space F_X is said to be (ν)-*complete* if the following condition is satisfied: if for a sequence $(f_n)_{n\in\mathrm{N}}$ of elements of F_X there exists a sequence $(x_n)_{n\in\mathrm{N}}$ of elements of X such that $x_n \underset{n\in\mathrm{N}}{\downarrow} \mathbf{0}$ and

$$\|f_i - f_j\|_X \le x_n,\ (i, j \ge n)$$

then the sequence $(f_n)_{n\in\mathrm{N}}$ (ν)-*converge* to an element of F_X.

In an Archimedean directed vector space Z, a sequence $(z_n)_{n\in\mathrm{N}}$ of elements is said [7] to *converge with regulator* (briefly (ρ)-*converge*) to an element $z \in Z$, if there exists an element $w \in Z$ and a sequence $(\varepsilon_n)_{n\in N}$ of positive numbers convergent to zero, such that

$$\pm(z_n - z) \le \varepsilon_n w, \ (\forall n \in N).$$

The element z is called the (ρ)-limit of the sequence $(z_n)_{n\in N}$.

If F_X is a v-normed vector space and if the vector lattice X is Archimedean, then a sequence $(f_n)_{n\in N}$ of elements of F_X is said to (νρ)-*converge* to an element $f \in F_X$, if in the relation (0.1) the order convergence is a convergence with regulator. We shall denote $f = (v\rho)\text{-}\lim_n f_n$.

If F_X and G_Y are v-normed vector spaces, a mapping $S: F_X \to G_Y$ is said to be (v)-*continuous* if from $f = (v)\text{-}\lim_n f_n$ in F_X, it follows $S(f) = (v)\text{-}\lim_n S(f_n)$.

We call a *strictly (v)-normed vector space*, a v-normed vector space F_X with the property: if $\|f\|_X = a_1 + a_2$ with $\mathbf{0} \le a_1, a_2 \in X$ then there exist $f_1, f_2 \in F_X$ such that $f = f_1 + f_2$ and $\|f_i\|_X = a_i, (i = 1,2)$.

1. VECTOR LATTICES WITH VECTOR NORM

Let F be a vector lattice endowed with a vector norm with values in a vector lattice X.

The vector norm of the space F_X is said to be *monotone* if $\mathbf{0} \le f_1 \le f_2$ in F_X implies $\|f_1\|_X \le \|f_2\|_X$. The vector norm is said to be *solid* if $|f_1| \le |f_2|$ in F_X implies $\|f_1\|_X \le \|f_2\|_X$.

It is easy to verify that the vector norm of the space F_X is solid if and only if it is a monotone vector norm which satisfies the condition

$$\| \, |f| \, \|_X = \|f\|_X, \ (\forall f \in F_X). \tag{1.1}$$

Proposition 1.1. If F is a vector lattice endowed with a vector norm with values in a vector lattice X, then the condition (1.1) is equivalent to the condition

$$f_1 \wedge f_2 = \mathbf{0} \text{ in } F_X \Rightarrow \| f_1 + f_2 \|_X = \| f_1 - f_2 \|_X . \tag{1.2}$$

Proof. If the condition (1.1) is verified and if $f_1 \wedge f_2 = \mathbf{0}$ in F_X then

$$| f_1 - f_2 | = f_1 + f_2$$

whence it follows

$$\|f_1 - f_2\|_X = \|f_1 + f_2\|_X .$$

Conversely if (1.2) holds then let $f \in F_X$ and let us consider the positive part f^+ of f and the negative part f^- of f. We have $f^+ \wedge f^- = 0$ and by (1.2) it follows

$$\| \, |f| \, \|_X = \| \, f^+ + f^- \, \|_X = \| \, f^+ - f^- \, \|_X = \| \, f \, \|_X$$

hence (1.1) holds.

Proposition 1.2. Let F be a vector lattice endowed with a solid vector norm with values in a vector lattice X. The following statements hold.

(i) The mapping $S : F_X \to F_X$ given by the formula $S(f) = |f|$ is (v)-continuous.

(ii) If $f' = (v)\text{-}\lim\limits_n f'_n$ and $f'' = (v)\text{-}\lim\limits_n f''_n$ and if $f'_n \le f''_n, \forall\, n \in \mathrm{N}$, then $f' \le f''$.

Proof. (i) We have only to take into account that

$$\| \, |f_1| - |f_2| \, \|_X \le \|f_1 - f_2\|_X, \ (\forall f_1, f_2 \in F_X)$$

since the vector norm is solid.

ii) we have

$$\left| f''_n - f'' \vee f'' \right| = \left| f'_n \vee f''_n - f' \vee f'' \right| \le \left| f'_n - f' \right| + \left| f''_n - f'' \right|$$

whence it follows

$$\|f''_n - f' \vee f''\|_X \le \|f'_n - f'\|_X + \|f''_n - f''\|_X$$

and, consequently

$$f' \vee f'' = (v)\text{-}\lim_n f''_n = f''$$

that is $f' \le f''$.

Definition 1.1. If F is a vector lattice endowed with a vector norm with values in a vector lattice X, we say that the space F_X has a *vector norm of type (M)* if

$$\mathbf{0} \le f_1, f_2 \in F_X \Rightarrow \|f_1 \vee f_2\|_X = \|f_1\|_X \vee \|f_2\|_X .$$

Example 1.1. Let X be a complete vector lattice, T a (nonvoid) set and let us denote by $B(T, X)$ the set of all (o)-bounded functions defined on T with values into X. With respect to the usual operations and the pointwise order, $B(T, X)$ is a complete vector lattice and the formula

$$\|f\|_X = \sup_{t \in T} |f(t)|, \ (f \in B(T, X))$$

defined a solid vector norm of type (M) on $B(T, X)$.

Proposition 1.3. Let F be a vector lattice endowed with a solid vector norm of type (*M*) with values in a vector lattice X. The following statements hold.

(i) If $(v)\text{-}\lim\limits_{n} f_n = \mathbf{0}$ in F_X, then for any sequence $(j_n)_{n\in\mathrm{N}}$ of natural numbers we have

$$(v)\text{-}\lim_{n} \bigvee_{i=0}^{j_n} |f_{n+i}| = \mathbf{0}. \tag{1.3}$$

(ii) If F_X is (v)-complete and if $(f_n)_{n\in\mathrm{N}}$ is a (v)-convergent sequence of elements of F_X, then there exist $\bigvee\limits_{n\in\mathrm{N}} f_n$ and $\bigwedge\limits_{n\in\mathrm{N}} f_n$.

Proof. (i) If $\|f_n\|_X \le v_n, (\forall n\in\mathrm{N})$, with $v_n \underset{n\in\mathrm{N}}{\downarrow} \mathbf{0}$, then

$$\left\| \bigvee_{i=0}^{j_n} |f_{n+i}| \right\|_X = \bigvee_{i=0}^{j_n} \|f_{n+i}\|_X \le v_n,\ (\forall n\in\mathrm{N})$$

whence it follows (1.3).

(ii) Let $f = (v)\text{-}\lim\limits_{n} f_n$ and $(v_n)_{n\in\mathrm{N}}$ a sequence of elements of X such that $v_n \underset{n\in\mathrm{N}}{\downarrow} \mathbf{0}$ and

$$\|f_n - f\|_X \le v_n,\ (\forall n\in\mathrm{N}).$$

By putting

$$g_n = \bigvee_{j=1}^{n} f_j,\ (n\in\mathrm{N})$$

we have for $m,n\in\mathrm{N}$ and $m<n$

$$\mathbf{0} \le g_n - g_m \le \bigvee_{i=m+1}^{n} |f_i - f_m|.$$

Since the vector norm of the space F_X is solid and of type (*M*), it follows

$$\|g_n - g_m\|_X \le \bigvee_{i=m+1}^{n} \|f_i - f_m\|_X$$

therefore

$$\|g_n - g_m\|_X \le 2v_m, (m<n).$$

Since the space F_X is (v)-complete, it exists $g = (v)\text{-}\lim\limits_{n} g_n$. By the Proposition 1.2 it follows

$$g = \bigvee_{n\in\mathrm{N}} g_n = \bigvee_{n\in\mathrm{N}} f_n.$$

The existence of the element $\wedge_{n\in\mathrm{N}} f_n$ one obtain by considering the sequence $(-f_n)_{n\in\mathrm{N}}$.

Definition 1.2. If F is a vector lattice endowed with a vector norm with values in a vector lattice X, we say that the space F_X has a *vector norm of type (L)*, if

$$\mathbf{0} \le f_1, f_2 \in F_X \Rightarrow \|f_1 + f_2\|_X = \|f_1\|_X + \|f_2\|_X.$$

Example 1.2. Let X be a σ-complete vector lattice and let us denote by $\ell(X)$ the set of all sequences $(x_n)_{n\in\mathrm{N}}$ of elements of X for which the series $\sum_n |x_n|$ is (o)-convergent. With respect to the usual operations and the pointwise order, $\ell(X)$ is a vector lattice and the formula

$$\|(x_n)_{n\in\mathrm{N}}\|_X = o\text{-}\sum_{n=1}^{\infty} |x_n|$$

defines a solid vector norm of type (L) on $\ell(X)$.

Proposition 1.4. Let F be a vector lattice endowed with a solid vector norm of type (L) with values in a σ-complete vector lattice X. If the space F_X is (v)-complete, then F is a σ-complete vector lattice.

Proof. Let $\mathbf{0} \le f_n \underset{n\in\mathrm{N}}{\uparrow}$, in F and $f_n \le f_0, (\forall n \in \mathrm{N})$. Since the vector norm of the space F_X is solid, we have

$$\|f_n\|_X \underset{n\in\mathrm{N}}{\uparrow}, \ \|f_n\|_X \le \|f_0\|_X, (\forall n \in \mathrm{N}).$$

On the other hand, we have for $n \ge m$

$$\|f_n\|_X = \|f_m + (f_n - f_m)\|_X = \|f_m\|_X + \|f_n - f_m\|_X.$$

hence

$$\|f_n - f_m\|_X = \|f_n\|_X - \|f_m\|_X, \ (n \ge m),$$

Since the space F_X is (v)-complete, there exists $f = (v)\text{-}\lim_n f_n$. By the proposition (1.2), from $f_n \le f_{n+i}$, $(\forall n, i \in \mathrm{N})$, it follows $f_n \le f$, $(\forall n \in \mathrm{N})$. By the same proposition, if $f_n \le f', (\forall n \in \mathrm{N})$, then $f \le f'$, hence $f = \vee_{n\in\mathrm{N}} f_n$.

2. OPERATORS WITH VALUES IN v-NORMED VECTOR SPACES

If X is an ordered vector space and G_Y a v-normed vector space, an operator $U : X \to G_Y$ is called an (ov)-*bounded* operator, if for any (o)-bounded subset A of X the set $U(A)$ is (v)-bounded.

An Archimedean vector lattice is said to be a *space* (o)-*bounded type* if the condition that a subset A be a (o)-bounded is equivalent to the condition:

$$a_n \in A,\ \lambda_n \in \mathrm{R},\ (n \in \mathrm{N}),\ \lim_n \lambda_n = 0 \Rightarrow (o)\text{-}\lim_n \lambda_n a_n = \mathbf{0}.$$

One can prove that if X is an Archimedean directed vector space and G_Y a v-normed vector space with Y a space of (o)-bounded type, then a linear operator $U : X \to G_Y$ in an (ov)-bounded operator, if and only if, from $(\rho)\text{-}\lim_n x_n = \mathbf{0}$ in X it follows $(v\rho)\text{-}\lim_n U(x_n) = \mathbf{0}.$

A more general result is given in the next theorem which generalizes also a result of C. Swartz [13]. Another generalization was given in [10].

Theorem 2.1. Let X be an Archimedean directed vector space, G_Y a v-normed vector space with Y an Archimedean vector lattice, and let $(U_n)_{n\in\mathrm{N}}$ be a sequence of linear operators which map X into G_Y. Consider the following two statements.

(i) For any (o)-bounded subset A of X, the set $\bigcup_{n\in\mathrm{N}} U_n(A)$ is (v)-bounded.

(ii) For any sequence $(j_n)_{n\in\mathrm{N}}$ of natural numbers and for any sequence $(x_n)_{n\in\mathrm{N}}$ of elements of X which (ρ)-converge to $\mathbf{0}$ it follows $(v\rho)\text{-}\lim_n U_{j_n}(x_n) = \mathbf{0}.$

The statement (i) implies the statement (ii), and if Y is a space of (o)-bounded type, then these statements are equivalent.

Proof. Suppose that the statement (i) holds and let $(\rho)\text{-}\lim_n x_n = \mathbf{0}$ in the space X. Let $(\alpha_n)_{n\in\mathrm{N}}$ be a sequence of natural numbers such that $\alpha_n \underset{n\in\mathrm{N}}{\uparrow} +\infty$ and such that the sequence $(\alpha_n x_n)_{n\in\mathrm{N}}$ be (o)-bounded [7]. Then there exists an element $y_0 \in Y$ such that for any sequence $(j_n)_{n\in\mathrm{N}}$ of natural numbers

$$\left\| U_{j_n}(\alpha_n x_n) \right\|_Y \le y_0,\ (\forall n \in \mathrm{N})$$

whence it follows $(v\rho)\text{-}\lim U_{j_n}(x_n) = \mathbf{0}.$

Suppose now that Y is a space of (o)-bounded type and that the statement (ii) holds. Let A be an (o)-bounded subset of the space X and let us denote

$$B = \bigcup_{j \in \mathrm{N}} U_j(A).$$

If $(g_n)_{n\in\mathrm{N}}$ is a sequence of elements of B, then for $n \in \mathrm{N}$ there exists $j_n \in \mathrm{N}$ and $x_n \in A$ such that $g_n = \mathrm{U}_{j_n}(x_n)$. If $(\lambda_n)_{n\in\mathrm{N}}$ is a sequence of real numbers convergent to zero, then we have $(\rho)\text{-}\lim_n \lambda_n x_n = \mathbf{0}$ and by (ii) it follows

$$(\rho)\text{-}\lim_n \lambda_n \|g_n\|_Y = \mathbf{0}.$$

Consequently the set

$$\{ \|g\|_Y \mid g \in B\}$$

is (o)-bounded, since Y is a space of (o)-bounded type.

Definition 2.1. Let X be a topological vector space (with the topology τ) and G_Y be a v-normed vector space. An operator $U : X \to G_Y$ is called a *(τv)-bounded operator* if for any (τ)-bounded subset A of X, the set $U(A)$ is (v)-bounded.

Remark 2.1. The set of all (τv)-bounded linear operators which map a topological vector space X into a v-normed vector space G_Y is a vector space with respect to the usual operations. If X is a topological ordered vector space [4], then any (τv)-bounded operator which map X into G_Y is an (ov)-bounded operator.

Definition 2.2. If X is a locally convex space and G_Y a v-normed vector space, an operator $U : X \to G_Y$ is called a *$(p\,v)$-bounded operator* if there exists a continuous seminorm p on X and an element $y_0 \in Y$ such that

$$\| U(x) \|_Y \le p(x) y_0, (\forall x \in X). \tag{2.1}$$

Theorem 2.2. If X is a locally convex space and if G_Y is a v-normed vector space with Y an Archimedean vector lattice, then a linear operator $U : X \to G_Y$ is a (pv)-bounded operator if and only if there exists a neighbourhood A of the origin in X such that the set $U(A)$ be (v)-bounded.

Proof. Let us assume that the operator U satisfies the condition (2.1) where p is a continuous seminorm on X. By putting

$$A = \{x \in X \mid p(x) \le 1\}$$

the set A is a neighbourhood of the origin in X and we have

$$\| U(x) \|_Y \leq y_0, \ (\forall x \in A) \tag{2.2}$$

therefore $U(A)$ is (v)-bounded.

Conversely, assume that there exists a neighbourhood A of the origin in X such that the set $U(A)$ be (v)-bounded. Then, let $y_0 \in Y$ satisfying (2.2). There exists a continuous seminorm p on X such that, denoting

$$B = \{x \in X \mid p(x) < 1\}$$

to have $B \subset A$.

For any number $\varepsilon > 0$ we have $(p(x)+\varepsilon)^{-1} x \in B, (\forall x \in X)$, hence

$$\| U(x) \|_Y \leq (p(x)+\varepsilon) y_0.$$

The space Y being Archimedean, it follows (2.1).

Remark 2.2. If X is a normed vector space and G_Y a v-normed vector space, a linear operator $U : X \to G_Y$ is a (τv)-bounded operator, if and only if there exists an element $y_0 \in Y$ such that

$$\|U(x)\|_Y \leq \|x\| y_0, (\forall x \in X). \tag{2.3}$$

In the next theorem, for a normed vector space X (τ the norm topology) and a v-normed vector space G_Y, an operator $U : X \to G_Y$ is said to be (τv)-*continuous*, if from $x = (\tau)\text{-}\lim\limits_n x_n$ in X it follows $U(x) = (v)\text{-}\lim\limits_n U(x_n)$.

Theorem 2.3. Let X be a normed vector space (τ the norm topology) and G_Y a v-normed vector space with Y a space of (o)-bounded type. If $U : X \to G_Y$ is a linear operator, then U is a (τv)-bounded operator, if and only if it is (τv)-continuous.

Proof. If U is a (τv)-bounded operator, then by Remark 2.2 there exists an element $y_0 \in Y$ satisfying (2.3). Since Y is an Archimedean space, by (2.3) it follows that U is (τv)-continuous.

Conversely, assume that the operator U is (τv)-continuous and let A be a (τ)-bounded subset of the space X. Denote

$$B = \left\{ \| U(x) \|_Y \;\middle|\; x \in A \right\}$$

and let $z_n \in B$, $(n \in \mathrm{N})$ hence $z_n = \|U(x_n)\|_Y$ with $x_n \in A$. Let $(\lambda_n)_{n \in \mathrm{N}}$ be a convergent to zero sequence of real numbers. Then we have $(\tau)\text{-}\lim\limits_n \lambda_n x_n = \mathbf{0}$ and since U is (τv)-continuous it follows $(v)\text{-}\lim\limits_n U(\lambda_n x_n) = \mathbf{0}$, hence $(o)\text{-}\lim\limits_n \lambda_n z_n = \mathbf{0}.$ Since the space Y is of (o)-bounded type, it follows that the set B is (o)-bounded. Consequently the operator U is $(\tau\nu)$-bounded.

Remark 2.3. Let X be a normed vector space (τ the norm topology), X_0 a (τ)-dense vector subspace of X and G_Y a (ν)-complete ν-normed vector space with Y an Archimedean vector lattice. If $U_0 : X_0 \to G_Y$ is a $(\tau\nu)$-bounded linear operator, then there exists a unique $(\tau\nu)$-bounded linear operator $U : X \to G_Y$ such that $U \mid X_0 = U_0$.

Indeed, if $x \in X$, then there exists a sequence $(x_n)_{n\in \mathrm{N}}$ of elements of X_0 such that $x = (\tau)\text{-}\lim_n x_n$. We shall define an operator $U : X \to G_Y$ by putting $U(x) = (\nu)\text{-}\lim_n U(x_n)$.

It is easily verified that the (ν)-limit exists and does not depend on the choice of the sequence $(x_n)_{n\in \mathrm{N}}$. The properties on the operator U are also easy to prove.

Let now $(f_n)_{n\in \mathrm{N}}$ be a sequence of elements of a ν-normed vector space F_X and let us denote

$$s_n = \sum_{j=1}^{n} f_j,\ (n \in \mathrm{N}).$$

If the sequence $(s_n)_{n\in \mathrm{N}}$ is (ν)-convergent and if $f = (\nu)\text{-}\lim_n s_n$, then we say that the series $\sum_n f_n$ is (ν)-convergent and we denote

$$f = \nu\text{-}\sum_{n=1}^{\infty} f_n.$$

Definition 2.3. Let F be a vector lattice endowed with a vector norm with values in a vector lattice X, and let G_Y be a ν-normed vector space with Y a σ-complete vector lattice. An operator $U : F_X \to G_Y$ is called a *summable operator* if it is linear and if for any sequence $(f_n)_{n\in \mathrm{N}}$ of elements of F_X for which the series $\sum_n |f_n|$ is (ν)-convergent it follows that the series $\sum_n \|U(f_n)\|_Y$ is (o)-convergent.

Theorem 2.4. Let F be a vector lattice endowed with a solid vector norm with values in a vector lattice X, and let G_Y be a v-normed vector space with Y a σ-complete vector lattice. If $U : F_X \to G_Y$ is a linear operator for which there exists a positive linear operator $V : F \to Y$ such that

$$\| U(f) \|_Y \leq V(|f|), (\forall f \in F) \tag{2.4}$$

then U is a summable operator.

Proof. Let $(f_n)_{n \in \mathrm{N}}$ be a sequence of elements of F_X for which the series $\sum_n |f_n|$ be (v)-convergent. By putting

$$h_n = \sum_{j=1}^{n} |f_j|, \; h = v\text{-}\sum_{j=1}^{\infty} |f_j|$$

we have $h_n \leq h_{n+m}$, $(\forall m, n \in \mathrm{N})$ and $h = (v)\text{-}\lim_m h_{n+m}$. By the Proposition 1.2 it follows $h_n \leq h, (\forall n \in \mathrm{N})$ hence

$$\sum_{j=1}^{n} \| U(f_j) \|_Y \leq \sum_{j=1}^{n} V(|f_j|) = V\left(\sum_{j=1}^{n} |f_j| \right) \leq V(h).$$

We infer that the series $\sum_n \| U(f_n) \|_Y$ is (o)-convergent, therefore U is a summable operator.

Remark 2.4. Let F_X and G_Y be as in the Theorem 2.4. Then an operator $U : F_X \to G_Y$ given by the formula

$$U(f) = \sum_{j=1}^{n} \varphi_j(f) g_j, \; (\forall f \in F_X)$$

with $g_j \in G_Y$ and φ_j positive linear functional on F_X is a summable operator.

Indeed, by putting

$$V(f) = \sum_{j=1}^{n} \varphi_j(f) \| g_j \|_Y, \; (\forall f \in F_X)$$

we obtain a positive linear operator $V : F \to Y$ for which the relation (2.4) is verified.

3. (ν)-REGULAR OPERATORS

If X and Y are ordered vector spaces, we shall denote by $\mathcal{R}(X,Y)$ the set of all regular operators [4] mapping X into Y. We shall denote also $X^r = \mathcal{R}(X,\mathbb{R})$.

Let F_X and G_Y two ν-normed vector spaces.

An operator $U : F_X \to G_Y$ is said [8] to be a (ν)-*bounded operator* if for any (ν)-bounded subset A of F_X, the set $U(A)$ is (ν)-bounded.

An operator $U : F_X \to G_Y$ is called [4] a (ν)-*regular operator* if it is additive and if there exists a positive additive operator $W : X \to Y$ such that

$$\|U(f)\|_Y \le W\left(\|f\|_X\right),\ (\forall f \in F_X). \tag{3.1}$$

Any (ν)-regular operator is a (ν)-bounded operator.

We shall denote by $\mathcal{V}(F_X, G_Y)$ the set of all (ν)-regular operators which map F_X into G_Y. This set is a vector space with respect to the usual operations.

If the space Y is Archimedean, then any positive additive operator $W : X \to Y$ is a linear operator and by (3.1) it follows that if $U \in \mathcal{V}(F_X, G_Y)$ and if Y is Archimedean then U is a linear operator.

If F_X is a strictly ν-normed vector space, if G_Y is a ν-normed vector space with Y a complete vector lattice, and if $U \in \mathcal{V}(F_X, G_Y)$, then there exists the smallest positive linear operator $W : X \to Y$ which satisfies (3.1) and such an operator will be denoted by $\|U\|_{\mathcal{R}}$. If $Y = \mathbb{R}$ we shall denote $\|U\|_{X^r}$.

It is easy to verify that if F_X, G_Y, H_Z are ν-normed vector spaces and if $U \in \mathcal{V}(F_X, G_Y)$ and $V \in \mathcal{V}(G_Y, H_Z)$, then $VU \in \mathcal{V}(F_X, H_Z)$. If F_X and G_Y are strictly ν-normed vector spaces and if Y and Z are complete vector lattices, then

$$\|VU\|_{\mathcal{R}} \le \|V\|_{\mathcal{R}} \bullet \|U\|_{\mathcal{R}}.$$

Remark 3.1. If F is a normed vector space (τ the norm topology) and G_Y is a ν-normed vector space, a linear operator $U : F_{\mathbb{R}} \to G_Y$ is a (ν)-regular operator, if and only if U is a $(\tau\nu)$-bounded operator.

Remark 3.2. Let F_X and G_Y be ν-normed vector spaces with X and Y Archimedean vector lattices and let $U : F_X \to G_Y$ be a linear operator. The following statements hold.

(i) If U is a (ν)-regular operator, then from $f = (\nu\rho)\text{-}\lim\limits_n f_n$ in the space F_X it follows $U(f) = (\nu\rho)\text{-}\lim\limits_n U(f_n)$

(ii) If Y is a space of (o)-boundedness type and if from $f = (v\rho)\text{-}\lim_n f_n$ in the space F_X it follows $U(f) = (v)\text{-}\lim_n U(f_n)$ then U is a (v)-bounded operator.

Theorem 3.1 Let F be a vector lattice endowed with a solid vector norm with values in a vector lattice X and let Y be a complete vector lattice in which the modulus is considered as vector norm. The following statements hold.

(i) If $U : F_X \to Y_Y$ is a (v)-bounded linear operator, then $U \in \mathcal{R}(F,Y)$.

(ii) If F_X is a strictly v-normed space and $U \in \mathcal{V}(F_X, Y_Y)$, then $|U| \in \mathcal{V}(F_X, Y_Y)$ and $\| |U| \|_{\mathcal{R}} = \|U\|_{\mathcal{R}}$.

Proof. (i) If A is an (o)-bounded subset of the space F, let $f_0 \in F$ such that $|f| \le f_0, \forall f \in A.$ Since the vector norm of the space F_X is solid, we have $\|f\|_X \le \|f_0\|_X, \forall f \in A,$ hence the set A is (v)-bounded. By hypothesis, the set $U(A)$ is (v)-bounded in Y_Y therefore (o)-bounded in the space Y. Since Y is a complete vector lattice, it results $U \in \mathcal{R}(F,Y)$.

(ii) There exists a positive linear operator $W : X \to Y$ such that

$$|U(f)| \le W\left(\|f\|_X\right), (\forall f \in F_X). \tag{3.2}$$

We have

$$|U|(|f|) = sup\left\{|U(f')| \,\big|\, |f'| \le |f|\right\}$$

and $|f'| \le |f|$ implies $\|f'\|_X \le \|f\|_X$ hence by (3.2) it follows

$$|U(f')| \le W\left(\|f\|_X\right)$$

therefore

$$|U|(|f|) \le W\left(\|f\|_X\right).$$

In particular

$$\big||U|(f)\big| \le W\left(\|f\|_X\right), (\forall f \in F_X) \tag{3.3}$$

hence $|U|$ is a v-regular operator.

As we have seen, a positive linear operator $W : X \to Y$ which satisfies the relation (3.2) satisfies also the relation (3.3). Conversely, since

$$|U(f)| \le |U|(|f|),\ (\forall f \in F)$$

and since the vector norm of the space F_X is solid, from (3.3) it follows (3.2). It results $\| |U| \|_{\mathcal{R}} = \| U \|_{\mathcal{R}}$.

The proof is complete.

If F_X is a v-normed vector space and if Y is an ordered vector space, an operator $U : F_X \to Y$ is said to be a (vo)-*bounded operator* if for any (v)-bounded subset A of the space F_X, the set $U(A)$ is (o)-bounded.

Theorem 3.2. Let F be a vector lattice endowed with a solid vector norm of type (M) with values in a vector lattice X. If Y is a space of type (KB), then any positive (v)-regular operator $U : F_X \to Y_{\mathrm{R}}$ is a (vo)-bounded operator.

Proof. It is known (see e.g. [4]) that in a space of type (KB) a subset B is (o)-bounded, if and only if the set

$$K = \left\{ \overset{n}{\underset{j=1}{\vee}} \left|y_j\right| \middle| y_j \in B;\ n \in \mathrm{N} \right\} \tag{3.4}$$

is norm bounded. Let A be a (v)-bounded subset of the space F_X and $x_0 \in X$ such that $\|f\|_X \le x_0, \forall f \in A.$

Let $\varphi : X \to \mathrm{R}$ a positive linear functional such that

$$\|U(f)\| \le \varphi\left(\|f\|_X\right), \left(\forall f \in F_X\right).$$

Denoting $B = U(A)$, let us consider the set K given by the formula (3.4). Let $z \in K$ hence

$$z = \overset{n}{\underset{j=1}{\vee}} \left|U\left(f_j\right)\right|$$

with $f_j \in A$. We have

$$\|z\| = \left\| \overset{n}{\underset{j=1}{\vee}} \left|U\left(f_j\right)\right| \right\| \le \left\| \overset{n}{\underset{j=1}{\vee}} U\left(\left|f_j\right|\right) \right\| \le \left\| U\left(\overset{n}{\underset{j=1}{\vee}} \left|f_j\right| \right) \right\| \le$$

$$\le \varphi\left(\left\| \overset{n}{\underset{j=1}{\vee}} \left|f_j\right| \right\|_X \right) = \varphi\left(\overset{n}{\underset{j=1}{\vee}} \left\|f_j\right\|_X \right) \le \varphi\left(x_0\right)$$

hence the set K is norm bounded. It follows that the set $U(A)$ is (o)-bounded, therefore U is a (vo)-bounded operator.

Remark 3.3. Let X be a σ-complete vector lattice with the vector norm given by the modulus, and let G_Y be a v-normed vector space with Y a σ-complete vector lattice. Let us consider the space $\ell(X)$ defined in the section 1

(Example 1.2) and let us consider also the vector space $\ell(G_Y)$ of all the sequences $(g_n)_{n\in \mathrm{N}}$ of elements of G_Y for which the series $\sum_n \|g_n\|_Y$ is (o)-convergent.

On the space $\ell(G_Y)$ we consider the vector norm given by the formula

$$\left\|(g_n)_{n\in \mathrm{N}}\right\|_Y = o\text{-}\sum_{n=1}^{\infty}\|g_n\|_Y .$$

If $U : X_X \to G_Y$ is a (v)-regular operator, then by the formula

$$h_U\left((x_n)_{n\in \mathrm{N}}\right) = \left(U(x_n)\right)_{n\in \mathrm{N}}, \left(\forall (x_n)_{n\in \mathrm{N}} \in \ell(X)\right)$$

we obtain a (v)-regular operator $h_U : \ell(X) \to \ell(G_Y)$.

Indeed, there exists a positive linear operator $V : X \to Y$ such that

$$\|U(x)\|_Y \le V(|x|),\ (\forall x \in X). \tag{3.5}$$

If $(x_n)_{n\in \mathrm{N}} \in \ell(X)$ then the series $\sum_n \|U(x_n)\|_Y$ is (o)-convergent since by (3.5) it results that U is a summable operator (Theorem 3.4) hence $h_U\left((x_n)_{n\in \mathrm{N}}\right) \in \ell(G_Y)$. By (3.5) it follows also that

$$\left\|h_U\left((x_n)_{n\in \mathrm{N}}\right)\right\|_Y \le V\left(\left\|(x_n)_{n\in \mathrm{N}}\right\|_X\right)$$

hence h_U, which is obviously linear is a (v)-regular operator.

Theorem 3.3. Let F be a vector lattice endowed with a solid vector norm of type (L) with values in a vector lattice X and let G_Y be a v-normed vector space with Y a σ-complete vector lattice. Then any (v)-regular operator $U : F_X \to G_Y$ is a summable operator.

Proof. Let $W : X \to Y$ be a positive linear operator which satisfies the condition (3.1).

By putting, for $\mathbf{0} \le f \in F$

$$V_0(f) = W\left(\|f\|_X\right)$$

we obviously have

$$V_0(f_1 + f_2) = V_0(f_1) + V_0(f_2), (0 \le f_1, f_2 \in F).$$

Putting now

$$V(f) = V_0(f^+) - V_0(f^-), (\forall f \in F)$$

(where f^+ is the positive part of f and f^- the negative part of f) we obtain a positive linear operator $V:F \to Y$ satisfying the inequality (2.4). By the Theorem 2.4, U is a summable operator.

In the sequel, if F_X is a v-normed vector space, we shall denote $(F_X)^* = \mathcal{V}(F_X, \mathrm{R})$.

By a remark given in [10], if F_X is a strictly v-normed vector space and if G_Y is a v-normed vector space with Y a complete vector lattice, then the map

$$P:\mathcal{V}(F_X, G_Y) \to \mathcal{R}(X,Y)$$

given by the formula $P(U) = \|U\|_{\mathcal{R}}$ is a vector norm.

If F_X is a strictly v-normed vector space, then we shall consider $(F_X)^*$ as a v-normed vector space.

Theorem 3.4. If F_X and G_Y are strictly v-normed vector spaces, with Y a complete vector lattice and if $U:F_X \to G_Y$ is a (v)-regular operator, then the formula

$$U^*(k) = kU, \left(\forall k \in (G_Y)^*\right) \tag{3.6}$$

defines a (v)-regular operator $U^*:(G_Y)^* \to (F_X)^*$.

Proof. Let us first observe that for $k \in (G_Y)^*$ we have $kU \in (F_X)^*$. Indeed, kU is obviously linear and it easily verified that

$$\left|(kU)(f)\right| \le \left(\|k\|_{Y^r} \bullet \|U\|_{\mathcal{R}}\right)\left(\|f\|_X\right),\ (\forall f \in F_X).$$

Therefore we have $U^*(k) \in (F_X)^*$ and

$$\left\|U^*(k)\right\|_{X^r} \le \|k\|_{Y^r} \bullet \|U\|_{\mathcal{R}}.$$

By putting

$$\Psi_U(\varphi) = \varphi\|U\|_{\mathcal{R}},\ \left(\forall \varphi \in Y^r\right)$$

we obtain a positive linear operator $\Psi_U : Y^r \to X^r$ and

$$\left\|U^*(k)\right\|_{X^r} \le \Psi_U\left(\|k\|_{Y^r}\right), \left(\forall k \in (G_Y)^*\right)$$

hence U^* is a (v)-regular operator, and the proof is finished.

Let now X and Y be vector lattices, Z a complete vector lattice and F_X, G_Y, H_Z v-normed vector spaces.

We shall denote by $\mathcal{R}(X \times Y, Z)$ the space of all regular operators [4] which map $X \times Y$ into Z.

We shall say that an operator $U : F_X \times G_Y \to H_Z$ is a (ν)-*regular operator* if is biadditive and if there exists a positive operator $W \in \mathcal{R}(X \times Y, Z)$ such that

$$\|U(f,g)\|_Z \le W(\|f\|_X, \|g\|_Y), (\forall f \in F_X, \forall g \in G_Y) \tag{3.7}$$

Since Z is an Archimedean space, the operator W is bilinear and by (3.7) it follows that U is also a bilinear operator.

We shall denote by $\mathcal{V}(F_X \times G_Y, H_Z)$ the set of all (ν)-regular operators which map $F_X \times G_Y$ into H_Z. With respect to the usual operations, this set is a vector space.

If $U \in \mathcal{V}(F_X \times G_Y, H_Z)$ then U is a (ν)-*bounded operator*, that is, if $A \subset F_X$ and $B \subset G_Y$ are (ν)-bounded sets, then $U(A \times B)$ is a (ν)-bounded set.

For any operator $U \in \mathcal{V}(F_X \times G_Y, H_Z)$ and any element $f \in F_X$, we shall consider the operator $U_f : G_Y \to H_Z$ given by the formula $U_f(g) = U(f,g), \forall g \in G_Y$.

Remark 3.4. The operator U_f is (ν)-regular. Indeed, first U_f is obviously a linear operator. Let $W : X \times Y \to Z$ a positive bilinear operator such that such (3.7) holds and let us consider the operator $S_f : Y \to Z$ given by the formula

$$S_f(y) = W(\|f\|_X, y), (\forall y \in Y). \tag{3.8}$$

We have

$$\|U_f(g)\|_Z \le S_f(\|g\|_Y), \ (\forall g \in G_Y) \tag{3.9}$$

and S_f is a positive linear operator. Hence $U_f \in \mathcal{V}(G_Y, H_Z)$.

In the next theorem, for a strictly ν-normed vector space G_Y and a ν-normed vector space H_Z, with Z a complete vector lattice, we shall consider the ν-normed vector space $\mathcal{V}(G_Y, H_Z)$.

Theorem 3.5. Let F_X and G_Y be strictly ν-normed vector spaces, H_Z a ν-normed vector space with Z a complete vector lattice and let $U \in \mathcal{V}(F_X \times G_Y, H_Z)$. Then the operator

$$T_U : F_X \to \mathcal{V}(G_Y, H_Z)$$

given by the formula $T_U(f) = U_f$ is a (ν)-regular operator, and the map

$$\Omega : \mathcal{V}(F_X \times G_Y, H_Z) \to \mathcal{V}(F_X, \mathcal{V}(G_Y, H_Z))$$

given by the formula $\Omega(U) = T_U$ is a vector isomorphism.

Proof. Let $W : X \times Y \to Z$ be a positive bilinear operator such that (3.7) holds and let us consider the operator $S_f : Y \to Z$ given by the formula (3.8). The operator T_U is obviously linear. By (3.9) we have

$$\|T_U(f)\|_{\mathcal{R}} \le S_f, (\forall f \in F_X)$$

hence for $y \in Y$ we have

$$\|T_U(f)\|_{\mathcal{R}}(y) \le W(\|f\|_X, y). \tag{3.10}$$

Now we define an operator $D : X \to \mathcal{R}(Y, Z)$ by the formula

$$(D(x))(y) = W(x, y), (\forall x \in X, \forall y \in Y).$$

We have $\mathbf{0} \le D \in \mathcal{R}(X, \mathcal{R}(Y, Z))$ and by (3.10) we obtain

$$\|T_U(f)\|_{\mathcal{R}} \le D(\|f\|_X), (\forall f \in F_X).$$

Hence T_U is a (v)-regular operator.

It is easy to see that Ω is a linear map. This map is also surjective. Indeed, if $V \in \mathcal{V}(F_X, \mathcal{V}(G_Y, H_Z))$ then by putting

$$U(f, g) = (V(f))(g), (\forall f \in F_X, \forall g \in G_Y)$$

we obtain a bilinear operator $U : F_X \times G_Y \to H_Z$. By putting now

$$W(x, y) = (\|V\|_{\mathcal{R}}(x))(y), (\forall x \in X, \forall y \in Y)$$

we obtain a positive bilinear operator $W : X \times Y \to Z$ for which the inequality (3.7) holds. Hence $U \in \mathcal{V}(F_X \times G_Y, H_Z)$ and $\Omega(U) = V$.

Obviously, if $\Omega(U) = \mathbf{0}$ then $U = \mathbf{0}$ hence Ω is also an injective map.

Consequently Ω is a vector isomorphism.

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