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Fuzzy Subgroups and Chains of Subgroups

Aurelian Claudiu VOLF
Faculty of Mathematics, "Al.I. Cuza" University, Iasi, Romania
e-mail: volf@uaic.ro

Abstract. The fuzzy subgroups of a group are classified up to a natural equivalence relation. The problem of counting all fuzzy subgroups of a finite group *G* (up to equivalence) is studied, which amounts to counting all chains of subgroups of *G*. The number of maximal chains of an arbitrary finite cyclic group is determined. A recurrence relation on *m* for the number of fuzzy subgroups of a cyclic group of order p^nq^m is given. Using this method, a computer algebra program produces a formula depending on *n* for any (reasonably small) *m*. The results and proofs here extend and clarify several topics studied in [8], [9].
Key words: fuzzy subgroup, chain of subgroups, lattice of subgroups.

1. Introduction

Several recent papers tackle the classification of the fuzzy subgroups of a group: 0, [3], [5], [6], [7], [8], [9]. An adequate equivalence relation on the set of all fuzzy subgroups of the group must be defined in order to obtain a meaningful classification. Such a relation is studied in this paper (similar relations were defined in [3], [5]). One can describe it as follows: two fuzzy subgroups are equivalent when they have the same set of level subgroups (see Theorem 6). Counting all classes of equivalence of fuzzy subgroups of a finite group *G* is shown to be equivalent to counting the chains of subgroups of *G* that terminate in *G*. The number of maximal chains of an arbitrary finite cyclic group is determined (Proposition 11). The results we obtain are in the case of a cyclic group *G* of order p^nq^m (*p*, *q* distinct primes): we find a formula (recurrent on *m*) for the number $\gamma_{n,m}$ of fuzzy subgroups of *G* (see Theorem 15 and formula (6)). This yields (via computer algebra software) explicit formulas in *n* for computationally accessible *m*. and answers a conjecture on $\gamma_{n,m}$ raised in [9]. The counting method we use is simpler and more transparent than in [8] and [9]. An open problem is to generalize the counting method to the case of a cyclic group of any order.

2. Fuzzy subgroups

Let *G* be a set and let $\mu : G \to [0,1]$ be a *fuzzy subset* of *G*. If $(G,\cdot)$ is a group, μ is called a *fuzzy subgroup* of *G* if, for any $x, y \in G$, $\mu(xy) \geq \min(\mu(x), \mu(y))$ and $\mu(x^{-1}) \geq \mu(x)$ (in fact, $\mu(x^{-1}) = \mu(x)$). Also, $\mu(e) = \max \mu(G)$, where *e* is the neutral element of G. For every $\alpha \in [0,1]$ define the *level subset* ${}_{\mu}G_{\alpha} = \{x \in G \mid \mu(x) \geq \alpha\}$.

A fundamental result is (see for instance [4]):

Proposition 1. *A fuzzy subset $\mu : G \to [0,1]$ of a group G is a fuzzy subgroup if and only if all level subsets of μ are subgroups (i.e. $\forall \alpha \in [0,1]$, ${}_{\mu}G_{\alpha}$ is a subgroup of G).*

We will sometimes write G_{α} instead of ${}_{\mu}G_{\alpha}$ if no confusion can occur. It is obvious that $\forall \alpha,\ \beta \in [0,\ 1]$, $\alpha \leq \beta$ implies $G_{\alpha} \supseteq G_{\beta}$. We write $\subset$, respectively $\supset$, for strict inclusion. If A is a set, $|A|$ denotes its cardinal.

For a given group G, the task of *classification* of fuzzy subgroups of G leads to the definition of a natural equivalence relation on the set of all fuzzy subsets of G. ([3], [5]). If $\mu,\ \nu$ are fuzzy subsets of G, then write $\mu \sim \nu$ if :

a) for any $x,\ \ y \in G$: $\mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y)$.

b) for any $x \in G$: $\mu(x) = 0 \Leftrightarrow \nu(x) = 0$.

This is indeed an equivalence relation. We use a (somewhat) weaker condition of fuzzy subset equivalence, dropping the condition b) above.

Definition 2. If $\mu,\ \nu$ are fuzzy subsets of G, define:

$$\mu \sim \nu \Leftrightarrow \left(\forall x,\ \ y \in G : \mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y)\right)$$

This approach is motivated by the Theorem 6, which shows this relation is closely related to the concept of level subgroup. First, we collect some basic facts.

Proposition 3. *Suppose G is finite and μ is a fuzzy subgroup of G. Then $\mu(G)$ is a finite set, say $\{\mu_1,\ldots,\ \mu_n\} \subset [0,\ 1]$, and suppose that $\mu_1 > \ldots > \mu_n$. For simplicity purposes, write ${}_{\mu}G_i$ instead of ${}_{\mu}G_{\mu_i}$. Then $\{e\} \subseteq {}_{\mu}G_1 \subset {}_{\mu}G_2 \subset \ldots \subset {}_{\mu}G_n = G$. Also, for any $x \in G$ and any $1 \leq i \leq n$:*

$$\mu(x) = \mu_i \Leftrightarrow i = \min\{j \mid x \in {}_{\mu}G_j\} \Leftrightarrow x \in {}_{\mu}G_i \setminus {}_{\mu}G_{i-1} \text{ (we set } {}_{\mu}G_0 = \varnothing\text{).}$$

Thus, $\mu(e) = \mu_1$. □

Lemma 4. Let $\mu,\ \nu$ be fuzzy subsets of G. Then $\mu \sim \nu$ if and only if for any $n \geq 2$ and any $x_1,\ x_2,\ldots,\ x_n \in G$,

$$\mu(x_1) > \mu(x_2) > \ldots > \mu(x_n) \Leftrightarrow \nu(x_1) > \nu(x_2) > \ldots > \nu(x_n).$$

Proof. Standard induction argument (the case $n=2$ is the definition of equivalence). □

Lemma 5. Let $\mu,\ \nu$ be fuzzy subsets of G with $\mu\sim\nu$. Then, for any $x,\ y\in G$, $\mu(x)=\mu(y)\Leftrightarrow\nu(x)=\nu(y)$.

Proof. Suppose $\mu(x)=\mu(y)$. We cannot have $\nu(x)>\nu(y)$, since this implies (using $\mu\sim\nu$) that $\mu(x)>\mu(y)$. Similarly, $\nu(x)<\nu(y)$ is impossible. □

Theorem 6. Let $\mu,\ \nu$ be fuzzy subsets of the set G. Then the following statements are equivalent:

i) $\mu\sim\nu$.

ii) There exists a strictly increasing bijection $\varphi:\operatorname{Im}\mu\to\operatorname{Im}\nu$ such that $\nu=\mu\circ\varphi$.

iii) There exists a strictly increasing bijection $\varphi:\operatorname{Im}\mu\to\operatorname{Im}\nu$ such that ${}_{\mu}G_{\alpha}={}_{\nu}G_{\varphi(\alpha)}$, for any $\alpha\in\operatorname{Im}\mu.$

iv) μ and ν have the same set of level subsets: $\{{}_{\mu}G_{\alpha}\mid\alpha\in\operatorname{Im}\mu\}=\{{}_{\nu}G_{\beta}\mid\beta\in\operatorname{Im}\nu\}$.

Proof. $i)\Rightarrow ii)$ Let $\alpha\in\operatorname{Im}\mu.$ There exists $x\in G$ such that $\alpha=\mu(x)$. Define $\varphi(\alpha)=\nu(x)$.

This definition of $\varphi(\alpha)$ is correct (it does not depend on the choice of $x\in G$ with $\alpha=\mu(x)$). Indeed, if $y\in G$ is such that $\mu(x)=\mu(y)$, then $\nu(x)=\nu(y)$ by the lemma. The mapping $\varphi:\operatorname{Im}\mu\to\operatorname{Im}\nu$ is strictly increasing: if $\alpha,\ \beta\in\operatorname{Im}\mu,$ $\alpha<\beta,$ then there exist $x,\ y\in G$ with $\mu(x)=\alpha,\ \mu(y)=\beta.$ By the definition of $\mu\sim\nu,$ we have $\nu(x)<\nu(y)$, that is, $\varphi(\alpha)<\varphi(\beta)$.

$ii)\Rightarrow iii)$ For any x we have:

$$x\in{}_{\mu}G_{\alpha}\Leftrightarrow\mu(x)\geq\alpha\Leftrightarrow\varphi(\mu(x))\geq\varphi(\alpha)\Leftrightarrow\nu(x)\geq\varphi(\alpha)\Leftrightarrow x\in{}_{\nu}G_{\varphi(\alpha)}.$$

$iii)\Rightarrow iv)$ $\{{}_{\nu}G_{\beta}\mid\alpha\in\operatorname{Im}\nu\}=\{{}_{\nu}G_{\varphi(a)}\mid\alpha\in\operatorname{Im}\mu\}=\{{}_{\mu}G_{\alpha}\mid\alpha\in\operatorname{Im}\mu\}.$

$iv)\Rightarrow i)$ For any $x,y\in G,$

$$\mu(x)>\mu(y)\Leftrightarrow y\in{}_{\mu}G_{\mu(y)}\setminus{}_{\mu}G_{\mu(x)}\Leftrightarrow y\in{}_{\mu}G_{\mu(y)}\setminus{}_{\mu}G_{\mu(x)}.$$ □

Corollary 7. Let $\mu,\ \nu$ be fuzzy subgroups of the finite group G. Then $\mu \sim \nu$ if and only if μ and ν have the same set of level subgroups (more precisely, $\{{}_{\mu}G_{\alpha} \mid \alpha \in [0,\ 1]\} = \{{}_{\nu}G_{\alpha} \mid \alpha \in [0,\ 1]\}$. □

Theorem 6 shows that *there is a bijection between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of G that terminate in G (the chains of the form $G_1 \subset G_2 \subset \ldots \subset G_n = G$, where all G_i 's are subgroups of G).*

Remark 8. *Clearly, Theorem* 6 *can be rephrased for fuzzy subsets of a finite set G (not necessarily fuzzy subgroups of a group G).*

3. Counting fuzzy subgroups and chains of subgroups

Theorem 6 and Proposition 3 show that the problem of counting *all fuzzy subgroups of a given finite group G* (up to the equivalence relation defined above) amounts to counting *all chains of subgroups that contain G*. This problem was considered in a series of papers: [5], [6], [7], [8], [9]. The most significant results were obtained in [9], but the formulas given there count *all* chains of subgroups of *G*, not only the chains ending in *G*.

Obviously, this problem depends entirely on $L(G)$ (the lattice of subgroups of *G*), and not on the group itself. This leads to a more general problem:

Problem 9. *Given a finite partially ordered set* $(L, \leq)$, *with greatest element* 1, *find (or count) all chains of elements of L that contain* 1 *(i.e., a chain of the form* $x_1 < x_2 < \ldots < x_n = 1,\ x_i \in L$ *).*

This is a very difficult problem in the general case. We study this problem in the case *L* is the lattice *L*(*G*) of subgroups of a finite cyclic group *G*. We write $|G|$ for the cardinal of *G*. *In what follows, G is a finite cyclic group unless specified otherwise.*

The following proposition summarizes some well known facts.

Proposition 10. *a) If* $|G| = n$, *then there is an order preserving bijection (in fact, a lattice isomorphism)* φ *between* $L(G)$ *and the lattice* D_n *of natural divisors of n (ordered by divisibility), given by* $\varphi(H) = |H|$, $\forall H \in L(G)$.

b) Let $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ *be the decomposition of n in distinct prime factors* p_1 , ... , p_k. *Let* $I_m = \{0,\ 1, \ldots,\ m\}$, *endowed with the natural order. On the Cartesian product* $I_{\alpha_1} \times I_{\alpha_2} \times \ldots \times I_{\alpha_k}$, *define the product order by*

$$(a_1,\ a_2, \ldots,\ a_k) \leq (b_1,\ b_2, \ldots,\ b_k) \quad \text{iff } a_i \leq b_i, \quad \forall i,\ 1 \leq i \leq k.$$

Then the lattice $(D_n, |)$ *is isomorphic to the lattice* $\left(I_{\alpha_1} \times I_{\alpha_2} \times \ldots \times I_{\alpha_k}, \leq\right)$ *by* $\psi : I_{\alpha_1} \times I_{\alpha_2} \times \ldots \times I_{\alpha_k} \to D_n$, $\psi\left(\left(a_1,\ a_2, \ldots,\ a_k\right)\right) = p_1^{a_1} \ldots p_k^{a_k}$. □

Given an ordered set $(L, \leq)$, a chain of elements of L is called a *maximal chain* if it is not properly included in another chain. Section 3 of the paper [8] studies the problem of counting the number of maximal chains of subgroups of a cyclic group. The following result completely solves this problem:

Proposition 11. *Assume* $p_1, \ldots,\ p_k$ *are distinct primes,* $\alpha_1, \ldots,\ \alpha_k$ *are positive integers and* G *is the cyclic group of order* $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$. *Then the number of maximal chains of subgroups of* G *is the multinomial coefficient* $\binom{\alpha_1 + \cdots + \alpha_k}{\alpha_1, \ldots,\ \alpha_k} = \dfrac{(\alpha_1 + \cdots + \alpha_k)!}{\alpha_1! \cdot \ldots \cdot \alpha_k!}$.

Proof. By Proposition 10, we count the maximal chains in $\left(I_{\alpha_1} \times I_{\alpha_2} \times \ldots \times I_{\alpha_k}, \leq\right)$. Write $\mathbf{c} = \left(c_1, \ldots,\ c_k\right)$ for elements of $I_{\alpha_1} \times I_{\alpha_2} \times \ldots \times I_{\alpha_k}$, and set $\mathbf{c}(i) = c_i$, $1 \leq i \leq k$. Then a maximal chain C has $m = \alpha_1 + \ldots + \alpha_k$ steps and C is of the form

$$C : (0, \ldots,\ 0) = \mathbf{c}_0 < \mathbf{c}_1 < \ldots < \mathbf{c}_m = \left(\alpha_1, \ldots,\ \alpha_k\right)$$

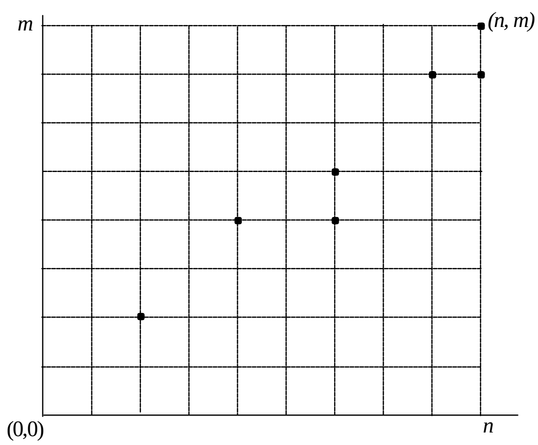
In such a chain, for every step s from 1 to m, $\mathbf{c}_s$ and $\mathbf{c}_{s+1}$ differ in exactly one component and the difference is 1: $(\exists!)$ i, $1 \leq i \leq k$, such that $\mathbf{c}_s(i) = \mathbf{c}_{s-1}(i) + 1$ and $\mathbf{c}_s(j) = \mathbf{c}_{s-1}(j)$ for all $j \neq i$. So, the chain C is uniquely determined by the sequence of the places in which consecutive k-uples differ: the mapping that associates to C the sequence $d = \left(d_s\right)_{1 \leq s \leq m}$, where d_s is the unique i with $\mathbf{c}_s(i) = \mathbf{c}_{s-1}(i) + 1$, is bijective. We see now that we must count all sequences of the type $\left(d_1, \ldots,\ d_m\right)$, where exactly α_1 of the d_i 's are 1 , α_2 of the d_i 's are 2 , …, α_k of the d_i 's are k. This is the number of arrangements of m objects in k boxes, such that the first box contains α_1 objects, the second contains α_2 objects,…, the k-th contains α_k objects. This number is precisely $\binom{m}{\alpha_1, \ldots,\ \alpha_k}$ (see for instance [10]). □

3.1. G is cyclic of order p^n

This case is simple. A chain of subgroups of G corresponds to a chain $0 \le x_1 < x_2 < \ldots < x_k = n,$ where $k \le n$ and $x_i \in I_n$. These chains are in bijection with the subsets of I_n that contain n and there are 2^n of those.

3.2. G is cyclic of order $p^n q^m$

This case has a geometric interpretation in plane. A chain in $(I_n \times I_m, \le)$ is an increasing path in the grid of points of integer coordinates with $0 \le x \le n,\ 0 \le y \le m.$ The path is increasing in the sense that the successor (c,d) of any point (a,b) of the path must satisfy $(a,b) < (c,d)$. Recall that we must count all chains in $I_n \times I_m$ that



contain (n,m).

Denote by $C_{n,\,m}$ the set of *all* non-empty chains in $I_n \times I_m$ and $F_{n,\,m}$ by the set of chains containing (n, m). Let $\gamma_{n,\,m} = \left|C_{n,\,m}\right|$ and $f_{n,\,m} = \left|F_{n,\,m}\right|$. We count first $\gamma_{n,\,m}$. For a chain $\mathbf{c} : \mathbf{c}_0 < \mathbf{c}_1 < \ldots < \mathbf{c}_k$ in $C_{n,\,m}$ and $\mathbf{c}_{k+1} > \mathbf{c}_k$, denote by $\mathbf{c} \circ \mathbf{c}_{k+1}$ the chain obtained by appending $\mathbf{c}_{k+1}$ to $\mathbf{c}$, i.e. $\mathbf{c} \circ \mathbf{c}_{k+1}$ is $\mathbf{c}_0 < \mathbf{c}_1 < \ldots < \mathbf{c}_k < \mathbf{c}_{k+1}$. If C is a set of chains, and $(a,\ b) \in I_n \times I_m$ is such that $(a,\ b)$ is strictly greater than any member of any chain in C, let

$$C\circ(a,\ b)=\{\mathbf{c}\circ(a,\ b)\mid \mathbf{c}\in C\}$$

Proposition 12. The following recurrence relation holds, for any $n,\ m>1$:

$$\gamma_{n,\ m}=1+2\gamma_{n,\ m-1}+2\gamma_{n-1,\ m}-2\gamma_{n-1,\ m-1} \quad (1)$$

Moreover, we have, for any $n,\ m\geq 1$:

$\gamma_{n,\ m}=\gamma_{m,\ n}$

$\gamma_{n,\ 0}=\gamma_{0,\ n}=2^{n+1}-1$

Proof We have (visualize on the grid):

$$C_{n,\ m}=\{(n,\ m)\}\dot{\cup}\left(C_{n,\ m-1}\cup C_{n-1,\ m}\right)\dot{\cup}\left[C_{n,\ m-1}\circ(n,\ m)\cup C_{n-1,\ m}\circ(n,\ m)\right] \quad (2)$$

where $\dot{\cup}$ means disjoint union (the terms of the union are disjoint sets).

Obviously, $\left|C_{n,\ m-1}\circ(n,\ m)\right|=\left|C_{n,\ m-1}\right|=\gamma_{n,\ m-1}$. Using the relation $C_{n,\ m-1}\cap C_{n-1,\ m}=C_{n-1,\ m-1}$ and the fact that $|A\cup B|=|A|+|B|-|A\cap B|$ (for any sets A and B), we obtain (1) by taking cardinals in (2). Since $(I_n\times I_m,\leq)$ is lattice isomorphic to $(I_m\times I_n,\leq)$ by the mapping $(a,\ b)\mapsto(b,\ a)$, we obtain $\gamma_{n,\ m}=\gamma_{m,\ n}$. Finally, $\gamma_{n,\ 0}$ is the number of non-empty chains in $I_n\times\{0\}$, which is simply the number of non-empty subsets in $I_n=\{0,\ 1,\ldots,\ n\}$; so $\gamma_{n,\ 0}=2^{n+1}-1$. □

This recurrence relation can be used for an explicit computation of $\gamma_{n,\ m}$. The connection between $\gamma_{n,\ m}$ and $f_{n,\ m}$ is given by:

Proposition 13 For any $n,\ m\geq 1$, we have:

$$f_{n,\ m}=\gamma_{n,\ m}-\gamma_{n-1,\ m}-\gamma_{n,\ m-1}+\gamma_{n-1,\ m-1} \quad (3)$$

$f_{n,\ 0}=f_{0,\ n}=2^n$.

Proof The relation follows by taking cardinals in

$F_{n,\ m}=C_{n,\ m}\setminus\left(C_{n,\ m-1}\cup C_{n-1,\ m}\right)$,

where

$$\left|C_{n,\ m-1}\cup C_{n-1,\ m}\right|=\left|C_{n,\ m-1}\right|+\left|C_{n-1,\ m}\right|-\left|C_{n,\ m-1}\cap C_{n-1,\ m}\right|=\left|C_{n,\ m-1}\right|+\left|C_{n-1,\ m}\right|-\left|C_{n-1,\ m-1}\right|.$$

Also, $f_{n,\,0}$ is the number of subsets of $I_n \times \{0\}$ that contain $(n,\ 0)$, which is 2^n.
□

Let us compute $\gamma_{n,\,1}$.

Proposition 14. For any $n \geq 1$, $\gamma_{n,\,1} = 2^n(2n+4)-1$.

Consequently, $f_{n,\,1} = 2^n(n+2)$.

Proof. We have, by Proposition 12:

$$\gamma_{n,\,1} = 1 + 2\gamma_{n,\,0} + 2\gamma_{n-1,\,1} - 2\gamma_{n-1,\,0} = 1 + 2\left(2^{n+1}-1\right) + 2\gamma_{n-1,\,1} - 2\left(2^n - 1\right)$$
$$= 1 + 2^{n+1} + 2\gamma_{n-1,\,1}$$

Write t_n instead of $\gamma_{n,\,1}$. So:

$$t_n = 1 + 2^{n+1} + 2t_{n-1}$$
$$2\cdot|\ t_{n-1} = 1 + 2^n + 2t_{n-2}$$
$$\ldots$$
$$2^{n-2}\cdot|\ t_2 = 1 + 2^3 + 2t_1$$
$$2^{n-1}\cdot|\ t_1 = 1 + 2^2 + 2t_0$$

By multiplying these equalities as indicated and adding them, we get:

$$t_n = 1 + 2 + \ldots + 2^{n-1} + n\cdot 2^{n+1} + 2^n t_0 = 2^n(2n+4)-1,$$

since $t_0 = 3$. □

This expression for $\gamma_{n,\,1}$ (and a similar one for $\gamma_{n,\,2}$) can be found in [9]. The proofs there involve direct counting in the lattice of subgroups.

For the general case, we prove first that $\gamma_{n,\,m}$ is of the form $2^n P_m(n)-1$, where P_m is a polynomial of degree *m* with rational coefficients and then try to compute $P_m(n)$. This answers a conjecture of [9], where formulas for $\gamma_{n,\,m}$ for $m \leq 4$ are given (without proof in the cases $m = 3$ and 4) and it was conjectured that $\gamma_{n,\,m}$ is a polynomial in *n*, of degree *m* (which is true), and also monic, which is clearly not true.

Theorem 15 *There exists a unique sequence of polynomials* $(P_m)_{m\geq 0}$ *with rational coefficients, such that, for any n, m* ≥ 0,

$$\gamma_{n,\,m} = 2^n P_m(n) - 1 \tag{4}$$

Moreover, $\deg P_m = m$ and

$$P_m(n) = P_m(n-1) + 2 \cdot P_{m-1}(n) - P_{m-1}(n-1) \quad (5)$$

Proof. The uniqueness assertion is clear: if two sequences $(P_m)_{m\geq 0}$ and $(Q_m)_{m\geq 0}$ satisfy the condition, then, for any $m \geq 0$, $P_m(n) = Q_m(n)$ for any $n \in \mathbb{N}$, so $P_m = Q_m$.

Proposition 14 shows that the existence part holds for $m \in \{0, 1\}$, with $P_0(n) = 2$, $P_1(n) = 2n + 4$. We prove the other claims by induction on m. By requiring that (4) is true and replacing the γ's in (1) with P's, we get (5).

Conversely, if this relation holds for any n and m, then $\gamma_{n,\, m} = 2^n P_m(n) - 1$ satisfies (1).

Write the formula (5) for n, $n-1$, …, 1 :

$$P_m(n) = P_m(n-1) + 2 \cdot P_{m-1}(n) - P_{m-1}(n-1)$$
$$P_m(n-1) = P_m(n-2) + 2 \cdot P_{m-1}(n-1) - P_{m-1}(n-2)$$
$$\ldots$$
$$P_m(1) = P_m(0) + 2 \cdot P_{m-1}(1) - P_{m-1}(0)$$

Adding the equalities above, we obtain:

$$P_m(n) = P_m(0) - P_{m-1}(0) + P_{m-1}(n) + \sum_{i=1}^{n} P_{m-1}(i)$$

Using that $\gamma_{0,\, m} = 2^0 P_m(0) - 1 = \gamma_{m,\, 0} = 2^{m+1} - 1$, we get $P_m(0) = 2^{m+1}$, so:

$$P_m(n) = 2^m + P_{m-1}(n) + \sum_{i=1}^{n} P_{m-1}(i) \quad (6)$$

By the induction hypothesis, we can write $P_{m-1}(X) = a_0 + a_1 X + \ldots + a_{m-1} X^{m-1}$, for some $a_i \in \mathbb{Z}$ with $a_{m-1} \neq 0$. Then (6) becomes:

$$P_m(n) = 2^m + a_0 \cdot n + P_{m-1}(n) + a_1 \sum_{i=1}^{n} i + \ldots + a_{m-1} \sum_{i=1}^{n} i^{m-1}$$

The sum $S_k(n)=\sum_{i=1}^{n} i^k$ is well-known to be expressible as a polynomial of degree $k+1$ in *n* with rational coefficients. This statement can easily be proven by induction on *k*, using the recurrence relation:

$$(n+1)^{k+1}=1+\binom{k+1}{1}\ S_k(n)+\binom{k+1}{2}\ S_{k-1}(n)+\ldots+\binom{k+1}{k}\ S_1(n)+n$$

So $P_m(n)=2^m+a_0\cdot n+a_1S_1(n)+\ldots+a_{m-1}S_{m-1}(n)+P_{m-1}(n)$ is a polynomial of degree *m* in *n* as an inspection of the degrees in *n* of the terms in the sum reveals. Its coefficients are rational. The relation

$$P_m(n)=2^m+P_{m-1}(n)+\sum_{i=1}^{n}P_{m-1}(i)$$

defines $P_m(n)$ if $P_{m-1}(n)$ is given; $P_m(n)$ thus defined satisfies (5), which achieves the proof of existence of the sequence $(P_m)_{m\geq 0}$. □

Explicit expressions for $P_m(n)$ can be found by applying the recurrence relation (6) (using a computer algebra program). We obtain:

$$P_1(n)=2n+4$$

$$P_2(n)=n^2+7n+8$$

$$P_3(n)=\frac{1}{3}n^3+5n^2+\frac{56}{3}n+16$$

$$P_4(n)=\frac{1}{12}n^4+\frac{13}{6}n^3+\frac{203}{12}n^2+\frac{269}{6}n+32$$

$$P_5(n)=\frac{1}{60}n^5+\frac{2}{3}n^4+\frac{107}{12}n^3+\frac{145}{3}n^2+\frac{1531}{15}n+64$$

$$P_6(n)=\frac{1}{360}n^6+\frac{19}{120}n^5+\frac{233}{72}n^4+\frac{713}{24}n^3+\frac{22637}{180}n^2+\frac{3377}{15}n+128$$

$$\ldots$$

$$P_{10}(n)=\tfrac{1}{1814400}n^{10}+\tfrac{31}{362880}n^9+\tfrac{47}{8640}n^8+\tfrac{2249}{12096}n^7+\tfrac{323413}{86400}n^6+\tfrac{793027}{17280}n^5+\tfrac{15547277}{45360}n^4$$
$$+\tfrac{27538801}{18144}n^3+\tfrac{2250299}{600}n^2+\tfrac{1442426}{315}n+2048$$

Accordingly, one obtains explicit formulas for $\gamma_{n,\,m}$ and $f_{n,\,m}$. The expressions obtained for $\gamma_{n,\,m}$ with $m \le 5$ agree with those in [9], where they are stated without proof for $m > 3$. The formulas for $m \ge 6$ are new, as far as I know.

Using symbolic computation software, it is relatively easy to write a program that, with *m* as an input, outputs an expression for $P_m(n)$ (and all $P_k(n)$ with $k < m$). Here is an example in Maple 6 (which produced the output above):

```
> restart; poly:=proc(m)
> if m=0 then 2
> else 2^m+poly(m-1)+sum(eval(poly(m-1),n=i),i=1..n);
> end if
> end;
> printf("Enter the degree of the polynomial :\n");
> m:=scanf(%e):
> for j from 1 to op(1,m) do
> p[j](n)=sort(collect(poly(j),n));
> end do;
```

An interesting phenomenon is that *all polynomials* P_m *with* $m \le 15$ *are irreducible* (again, this was determined using Maple). It is an open problem if this is true for all *m*.

The method of counting described here can be carried to the case of a cyclic group *G* having 3 (or more) distinct prime divisors, for instance $|G| = p^a q^b r^c$. One can easily obtain a recurrence relation similar to (1) for the number $\gamma_{a,\,b,\,c}$ of chains of subgroups of *G*, but the actual computations for explicit formulas are more difficult.

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