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Similitude Relations and Applications in Social Sciences[1]

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Abstract. In this paper we consider some definitions and results about fuzzy operations and fuzzy relations. In particular we consider the similitude relations and the fuzzy partition. We show some applications of these concepts to some problems of Social Sciences and Town-Planning.

Keywords. Fuzzy relations, fuzzy partitions, decision making problems in Social Sciences, classifications in Town-planning.

The first records of statistic methods appear in the seventeenth century, when John Graunt (1620 - 1674) together with W. Petty have invented the "political arithmetic" as method of study of the social phenomena using numbers and measures. In the same time, in Germany, the descriptive statistics develops as the analysis of the economic resources, of the commercial development, interconnected with population growth.

1. Introduction

We recall that a *De Morgan algebra* is a lattice (L, $\wedge$, $\vee$) having 0 and 1 (respectively initial and final element) and, for which an operator
$\neg$: L$\to$L is defined such that:

(DMA) $\forall x, y \in L, \neg(x \wedge y) = (\neg x) \vee (\neg y), \neg(\neg x) = x, \neg 1 = 0.$

Example 1.1. Let L be the interval [0, 1] of an ordered field K and let $\wedge$ = min, $\vee$ = max and $\neg x = 1-x$. We have that (L, $\wedge$, $\vee$, $\neg$, 0, 1) is a De Morgan algebra.

Example 1.2. Let K be an ordered field K, L = {0; 0,1;…; 0,9; 1} $\subseteq$ K and let $\wedge$ = min, $\vee$ = max and $\neg x = 1-x$. Then (L, $\wedge$, $\vee$, $\neg$, 0, 1) is a De Morgan algebra.

We can remark also that, on the example above, L is not a Boole algebra.

By *fuzzy set with universe X* on a De Morgan algebra L (or L-fuzzy set) we mean a couple (X, μ) where X is a non-void set and μ: X$\to$L. We denote with

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$\Im_L(X)$ the family of all the fuzzy sets on L with universe X. For every $A \subseteq X$, we shall denote χ_A: $X \to L$ the characteristic function of A, that is the function such that $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \in X$-A. We denote with $\varnothing$ the characteristic function of the empty set $\varnothing$.

In the following we assume that L is a subset of an ordered field F and 0 and 1 are the zero and the unity of F. In particular we can have F=R and L = [0, 1] or L = {0; 0,1;…; 0,9; 1}.

In $\Im_L(X)$ the following operations will be considered:

- "$\curlywedge$" defined by $(\mu \curlywedge \nu)(x) = \min \{\mu(x), \nu(x)\}$;
- "$\curlyvee$" defined by $(\mu \curlyvee \nu)(x) = \max \{\mu(x), \nu(x)\}$;
- "$\odot$" defined by $(\mu \odot \nu)(x) = \max \{0, \mu(x) + \nu(x) - 1\}$;
- "$\oplus$" defined by $(\mu \oplus \nu)(x) = \min \{1, \mu(x) + \nu(x)\}$;
- "$\boxdot$" defined by $(\mu \boxdot \nu)(x) = \mu(x)\, \nu(x)$;
- "$\boxplus$" defined by $(\mu \boxplus \nu)(x) = \mu(x) + \nu(x) - \mu(x)\, \nu(x)$;
- "$'$" defined by $\mu'(x) = 1 - \mu(x)$.

It is known that the binary operations $\curlywedge$, $\curlyvee$, $\odot$, $\oplus$, $\boxdot$, $\boxplus$ are associative and commutative operations.

We remark also that the binary operations defined above can be extended to finite or countable family of fuzzy sets. Let $(\mu_i)_{i \in I}$ a finite or countable family of elements of $\Im_L(X)$. We have:

$$\curlywedge_{i \in I}\mu_i(x) = \inf \{\mu_i(x), i \in I\}; \qquad \curlyvee_{i \in I}\mu_i(x) = \sup \{\mu_i(x), i \in I\};$$

$$\odot_{i \in I}\, \mu_i(x) = \max \{0, 1 - \textstyle\sum_{i \in I}\mu'_i(x)\}; \qquad \oplus_{i \in I}\mu_i(x) = \min \{1, \textstyle\sum_{i \in I}\mu_i(x)\}.$$

2. Similarity relations

Let ρ: $X \times X \to L$, where L is a De Morgan algebra. We say that ρ is a fuzzy relation on X with range L or L-relation on X. We denote by $\Re_L(X)$ the set of L-relations on X. We define the *diagonal* Δ: $X \times X \to L$ by $\Delta(x, y) = 0$ if $x \neq y$ and, $\forall x \in X$, $\Delta(x, x) = 1$. The *inverse* ρ^{-1} of ρ is the L-relation on X such that, $\forall x, y \in X$, $\rho^{-1}(x, y) = \rho(y, x)$.

We assume, $\forall \rho \in \Re_L(X)$, $\rho^0 = \Delta$.

For any $\rho_1, \rho_2 \in \Re_L(X)$, we define $\rho_1 \bullet \rho_2$: $X \times X \to L$ by

$$(\rho_1 \bullet \rho_2)(x, y) = \vee_{z \in X}\, (\rho_1(x, z) \wedge \rho_2(z, y)).$$

We denote, $\forall n \in N$, $n > 0$, $\rho^n = \rho^{n-1} \bullet \rho$.

Definition 2.1. The fuzzy relation ρ is said to be:

(FR1) *reflexive*, if, $\forall x\in X$, $\rho(x, x) = 1$;
(FR2) *symmetric*, if, $\forall x, y\in X$, $\rho(x, y) = \rho(y, x)$;
(FR3) *antisymmetric*, if, $\forall x, y\in X$, $(\rho(x, y) = \rho(y, x)) \Rightarrow x = y$;
(FR4) *z-transitive*, if, $\forall x, y\in X$, $\rho(x, y) \geq \vee_{t\in X}(\rho(x, t)\wedge\rho(t, y))$.

For any $\rho, \sigma\in\Re_L(X)$, we put $\rho \subseteq \sigma$ if $\rho(x, y) \leq \sigma(x, y)$, $\forall\ x, y\in X$.

Remark 2.2. The fuzzy relation ρ is:

- reflexive is and only if $\Delta \subseteq \rho$;
- symmetric if and only if $\rho = \rho^{-1}$;
- z-transitive if and only if $\rho \supseteq \rho^2$.

In particular, for any fuzzy relation ρ, the relation $\rho^* = \rho \vee \Delta \vee \rho^{-1}$ is reflexive and symmetric.

Definition 2.3. A reflexive, symmetric and z-transitive L-relation on X is called a similarity relation on X.

Proposition 2.4. A L-relation ρ on X is a similarity relation on X if and only if, $\forall\alpha\in L$, the set

$$_{\rho}(X\times X)_{\alpha} = \{(x, y)\in X\times X/\ \rho(x, y) \geq \alpha\}$$

is an equivalence relation on X.

$$\rho_s = \vee_{n\in N}(\rho\vee\Delta\vee\rho^{-1})^n$$

is the smallest similarity relation such that $\rho \subseteq \rho_s$.

Proof. It is a consequence of the fact that the composition of two reflexive/symmetric relations is a reflexive/symmetric relation.

Proposition 2.6. *If $\rho\in\Re_L(X)$ is a similarity relation, then, for every x, y, z$\in$X, we have:*

$$\rho(x, y) = \rho(y, z) \text{ or } \rho(x, z) = \rho(y, z) \text{ or } \rho(x, z) = \rho(x, y).$$

Proof. If the assertion is not valid, we can suppose, without loss of generality, that $\rho(x, y) < \rho(x, z)<\rho(y, z)$. Then

$$\rho(x, y)\geq\ \vee_{t\in X}(\rho(x, t) \wedge \rho(t, y)) \geq \min\{\rho(x, z), \rho(z, y)\},$$

is a contradiction.

3. Partitions

Let J = {1, 2, …, n}.

Definition 3.1. The fuzzy sets $\mu_1, \mu_2, \ldots, \mu_n \in \Im_L(X)$ are called:

- *s-disjoint*, if, $\forall k \in J$, $(\oplus_{i\in J-\{k\}} \mu_j) \odot \mu_k = \varnothing$;
- *w-disjoint*, if $\odot_{i=1,\ldots,n} \mu_j = \varnothing$;
- *i-disjoint*, if, $\forall r, s \in J$, $r \neq s$, $\mu_r \wedge \mu_s = \varnothing$;
- *t-disjoint*, if, $\forall r, s \in J$, $r \neq s$, $\mu_r \boxdot \mu_s = \varnothing$.

We say that the letters s, w, i, t are *associated*, respectively, to the operations $\oplus$, $\oplus$, $\vee$, $\boxplus$.

Remark 3.2. The above definitions can be extended in a natural manner to a countable family of fuzzy set of $\Im_L(X)$: $\forall \alpha \in \{s, w, i, t\}$, $\mu_1, \mu_2, \ldots, \mu_n, \ldots \in \Im_L(X)$ are α-disjoint if, for any $n \in N$, $\mu_1, \mu_2, \ldots, \mu_n$ are α-disjoint.

Remark 3.3. We have:

(P1) If $\mu_1 \wedge \mu_2 = \varnothing$ then $\mu_1 \odot \mu_2 = \varnothing$. The converse is not generally true. It is true if μ_1 and μ_2 are characteristic functions.

(P2) $\mu_1 \odot \mu_2 = \varnothing \Leftrightarrow (\mu_1 \oplus \mu_2)(x) = \mu_1(x) + \mu_2(x)$, $\forall x \in X$.

(P3) Let $(A_i)_{i\in J}$ a family of n subsets of X and let χ_i the characteristic function of A_i, $i \in J$. The χ_i, $i \in j$, are s-disjoint if and only if, $\forall i, j \in J$, $i \neq j$, $\chi_i \odot \chi_j = \varnothing$.

(P4) $\mu_1 \wedge \mu_2 = \varnothing$ if and only if $\mu_1 \boxdot \mu_2 = \varnothing$.

(P5) $\mu_1, \mu_2, \ldots, \mu_n$ are s-disjoint $\Rightarrow$ $\mu_1, \mu_2, \ldots, \mu_n$ are w-disjoint.

Proposition 3.4. *We have:*

- $\mu_1, \mu_2, \ldots, \mu_n$ *are s-disjoint* $\Leftrightarrow \forall x \in X$, $\mu_1(x) + \mu_2(x) + \ldots + \mu_n(x) \leq 1$;
- $\mu_1, \mu_2, \ldots, \mu_n$ *are s-disjoint* $\Leftrightarrow \forall x \in X$, $\Sigma_{i\in J} \mu_i(x) = \oplus_{i\in J} \mu_i(x)$;
- $\mu_1, \mu_2, \ldots, \mu_n$ *are w-disjoint* $\Leftrightarrow \forall x \in X$, $\mu'_1(x) + \mu'_2(x) + \ldots + \mu'_n(x) \leq 1$;
- $\mu_1, \mu_2, \ldots, \mu_n$ *are w-disjoint* $\Leftrightarrow \forall x \in X$, $\mu_1(x) + \mu_2(x) + \ldots + \mu_n(x) \leq n\text{-}1$.

Correspondently we obtain the notion of σ-partition with $\sigma \in \{s, w, i, t\}$.

Definition 3.5. Let σ be an element of {s, w, i, t} and let Φ be the associated operation. The family $\{\mu_i\}_{i\in J} \subseteq \Im_L(X)$ is called a σ-partition of $\mu \in \Im_L(X)$ if $\mu_1, \mu_2, \ldots, \mu_n$ are σ-disjoint and $\Phi_{i\in J} \mu_i = \mu$.

Similarly one can define the countable partitions of a fuzzy subset of X. When $\mu = \chi_A$ with A subset of X, the σ-partition is called *fuzzy σ- partition* of A.

Remark 3.6 If $\{\mu_1, \mu_2, \ldots, \mu_n\}$ is a s-partition of μ and $\nu \leq \mu$, then $\{\nu \boxdot \mu_1, \nu \boxdot \mu_2, \ldots, \nu \boxdot \mu_n\}$ is a s-partition for $\nu \boxdot \mu$.

Let ρ: $X \times X \rightarrow L$ be a non-degenerate similarity relation (that is there exist x, $y \in X$, $x \neq y$, such that $\rho(x, y) = 1$). In the following we consider that X is a finite or countable set.

For $x \in X$ we denote μ_x: $X \rightarrow L$ the function such that $\mu_x(y) = 1$ if $\rho(x, y) = 1$ and $\mu_x(y) = 0$ if $\rho(x, y)$ is different from 1.

Proposition 3.7 If $\exists z \in X$ such that $\mu_x(z) = \mu_y(z) = 1$, then $\mu_x = \mu_y$.

Proof. We have: $\mu_x(z) = \mu_y(z) = 1 \Rightarrow \rho(x, z) = 1 = \rho(y, z)$. The property of z-transitivity implies that $\rho(x, y) = 1$. The same property implies: $\forall u \in X$, $\rho(x, u) = \rho(y, u)$. Then $\mu_x = \mu_y$.

The relation on X, x~y if and only if $\mu_x = \mu_y$ is an equivalence relation on X. Let $K = X/\sim$ and denote by [x] the class of x. Define $\mu_{[x]} = \mu_x$.

Proposition 3.8 The set $H = \{\mu_{[x]}, x \in X\}$ is a fuzzy w-partition and a fuzzy i-partition of X.

Proof. We have that $\Sigma_{\alpha \in K}\, \mu_\alpha(x) \leq n\text{-}1$, $\forall x \in X$, where n = card K.

It is also clear that $\oplus_{\alpha \in K}\, \mu_\alpha = \chi_X$.

We have $\forall \alpha, \beta \in K$, $\mu_\alpha \wedge \mu_\beta = =$. Evidently $\vee_{\alpha \in K}\, \mu_\alpha = \chi_X$.

4. Applications to Social Sciences and Town-Planning

In many problems of Social Sciences and Town-Planning it is very important to consider some relations in order to obtain classifications and orders in the set of the objects that must be studied. Such relations are, in general, fuzzy relations.

A first problem in this contest is the comparison among alternative projects.

If we have many projects, we need to consider a set $\Omega = \{O_1, O_2, \ldots, O_n\}$ of *objectives* and a set $K = \{C_1, C_2, \ldots, C_k\}$ of *criteria.*

For any objective O_i and for any criterion C_j we have to consider a real number w_{ij} belonging to a lattice L (usually L = [0, 1] or L = {0; 0,1;…; 0,9; 1}) that represents the weight of the criterion C_i as the objective O_j. Any criterion C_j defines the fuzzy set μ_j, with universe Ω, such that $\mu_j(O_i) = w_{ij}$.

The fundamental condition to compare weights of different objectives is that the fuzzy sets μ_j are a s-partition of K.

If $H = \{P_1, P_2, \ldots, P_c\}$ is the set of projects to compare, we have to consider, for any objective O_i, criterion C_j and project P_r a real number a_{ij}^r belonging to L that represents the score of the project P_r with respect to the criterion C_i and the objective O_j. For every project P_r, any criterion C_j defines the fuzzy set α_j^r, with universe Ω, such that $\alpha_j^r(O_i) = a_{ij}^r$.

In our paper [4] is considered, $\forall j \in \{1, 2,\dots, k\}$ the "product" $\nu_j^r = \mu_j * \alpha_j^r$ defined by the usual multiplication as $\nu_j^r(O_i) = \mu_j(O_i)\ \alpha_j^r(O_i)$, $\forall i \in \{1, 2,\dots, n\}$ and the fuzzy set ν_j^r is called *the utility* of the project P_r as to the criterion C_j. The utilities ν_j^r are s-disjoint and their "sum" with the operation $\oplus$ is a fuzzy set ν^r that represents the total fuzzy utility of P_r.

We can have a different type of utility γ^r, with particular meaning, if we sum the ν_j^r with any other fuzzy operation considered in this paper. If we consider a fuzzy set ω with universe Ω such that, $\forall i \in \{1, 2,\dots, n\}$, $\omega(O_i)$ is the measure of the importance of O_i, any function f: $\Im_L(\Omega)\times\Im_L(\Omega)\rightarrow L$ (with L = [0, 1] or L = {0; 0,1;…; 0,9; 1}), increasing as to any variable, can be utilized to calculate a scalar utility $u(P_r) = f(\gamma^r, \omega)$ of the project P_r.

A second actual problem (see [5], [6]) is the determination of the microzones in a city from the point of view of the rents, the sales and of the taxation of the houses. The new laws impose a classification of the city in microzones such that equivalent buildings relatively to a set K of characteristics must be in the same microzone. A hard classification, based on a crisp partition of the set X of buildings, is not suitable because it can assign very different rents or sales to near houses. So it is necessary to consider fuzzy partitions of X.

Therefore, in a set X of objects regarding Architecture and Town-Planning (the buildings or the projects) some reflexive and symmetric relations are introduced, but, unfortunately, in general these relations are not transitive. In fact, if $x_1, x_2,\dots,x_n$ are objects such that, for i = 1, 2, …, n-1, x_i and x_{i+1} have close characteristics, we can have, in general, that the characteristics of x_1 and x_n are very different. So, in general, we cannot classify X with a crisp method by these relations.

We shall consider a fuzzy classification of X starting with fuzzy relations.

Let L = [0, 1] or L = {0; 0,1;…; 0,9; 1}. An equivalence relation on X can be defined as an application ρ: $X\times X \rightarrow L$ such that:

(E1) $\forall x\in X$, $\rho(x, x) = 1$;

(E2) $\forall x, y\in X$, $\rho(x, y) = \rho(y, x)$;

(E3) $\forall x, y, z\in X$, $\rho(x, z) = \sup_{y\in X}\{\min\{\rho(x, y), \rho(y, z)\}\}$;

(E4) $\forall x, y\in X$, $\rho(x, y) = 0$ or $\rho(x, y) = 1$.

There are many possible generalizations, in the fuzzy frame, of the notion of equivalence relation. The conditions that, intuitively, appear to be necessary are the following:

(C1) in a fuzzy ambit we don't consider (E4) since two objects x and y are not necessarily in *null* relation ($\rho(x, y) = 0$) or in *complete* relation ($\rho(x, y) = 1$) but it is possible a *partial* relation ($0<\rho(x, y)<1$). The number $\rho(x, y)$ is called "the *degree* of relation ρ between x and y";

(C2) we must define an operation *, in L commutative and associative, such that, $\forall a, b\in[0, 1]$, a*b is not superior to the minimum between a and b. For any triplet (x, y, z) is defined as "degree of relation ρy between x and z way y", noted as $\rho y(x, z)$, the "product" $\rho(x, y)*\rho(y, z)$;

(C3) we must define an operation ◊, in L, associative and commutative, such that, $\forall a, b\in[0, 1]$, a◊b is not inferior to the maximum between a and b. If $X = \{y_1, y_2, \ldots, y_n\}$, $\rho(x, z)$ is at least equal to the "sum" $\rho^{y1}(x, z) \lozenge \rho^{y2}(x, z) \lozenge \ldots \lozenge \rho^{yn}(x, z)$.

These conditions lead us, in a natural way, to represent the problems of Social Sciences or Town-Planning with a similitude relation on X. By proposition 2.4 it follows that by a similitude relation ρ on X we obtain the set $\rho(X\times X)\alpha = \{(x, y)\in X\times X / \rho(x, y) \geq \alpha\}$, $\alpha\in L$ of equivalence relations on X. So, if we put,

$$\forall(x, y)\in X\times X,\ x\rho_\alpha y \text{ if and only if } (x, y)\in{}_\rho(X\times X)_\alpha,$$

the similitude relation ρ is equivalent of a set of equivalence relations $\{\rho_\alpha\}_{\alpha\in L}$ and any ρ_α is an approximation of ρ by a particular point of view.

Practically we can assume that L is substituted with the set

$$L^* = (L-\{0\})\cup\{0^+\}.$$ [2]

Let us consider the problem of a partition of a city in microzones. If ρ is a similitude relation in the set of buildings, then, for $\alpha\in L^*$, we have a set $\{K_\alpha\}_{\alpha\in S}$ of card(L*) crisp classifications of the city, that we call "*classifications associated* to ρ". In particular, if L = {0; 0,1;…; 0,9; 1}, we have 11 crisp classifications associated to ρ.

We have that, $\forall\alpha, \beta\in L$, $\alpha<\beta \Rightarrow$ any element of K_α is a union of elements of K_β, and so $\{K_\alpha\}_{\alpha\in L}$ is a hierarchical classification starting by K_{0+} as root.

We assume that a set of experts on Town-Planning agree on a reflexive and symmetric fuzzy relation ρ on X. We say that ρ is a quasi similitude relation given by the experts. We say that the assessment of ρ is "coherent" if the z-transitivity holds, that is ρ is a similitude relation.

By proposition 2.5 we have a general way to obtain by any relation ρ a similitude relation that is the z-transitive closure of ρ. If ρ is a quasi similitude, we can have the z-transitive closure ρ_s of ρ by considering $\rho_s = \vee_{n\in N} \rho^n$. Practically, we obtain ρ_s in a finite number of steps by the recurrent formula $\rho_{s,\,n+1} = \rho_{s,\,n} \vee \rho$. We can fix a small positive number ε and stop the algorithm if we have,

$$\forall x, y\in X,\ \rho_{s,\,n}(x, y) \geq \rho_{s,\,n}(x, t) \wedge \rho_{s,\,n}(t, y) - \varepsilon,\ \forall t\in X.$$

[2] We assume ${}_\rho(X\times X)_{0+} = \{(x, y)\in X\times X / \rho(x, y) > 0\}$, that is ${}_\rho(X\times X)_{0+}$ is the support of ρ.

If we have obtained a similitude relation ρ, we can classify X in many ways. Suppose we wish obtain a classification in about c classes. We find the number $\alpha \in L$ such that the number of classes of the equivalence relation ρ_α is nearest to c. If $\alpha<1$, we fuzzify the partition obtained, by considering the relations ρ_β such that $\beta>\alpha$. We consider a decreasing function f: $[0, 1]\to[0, 1]$ such that $f(t) = 1$ for $t \leq \alpha$.

For every x, $y \in X^2$, such that $t = \inf \{\beta \in [0, 1]: x\rho_\beta y\}$ we put

$$s(x, y) = f(t).$$

If [x] is the class of ρ_α containing the element x, we put,

$$\forall y \in X,\ s_{[x]}(y) = \max \{s(x, y), x \in [x]\}.$$

Let Π the set of the classes. $\forall y \in X$, we put

$$s(y) = \Sigma_{c\in\Pi}\, s_c(y) \text{ and } \mu_{[x]}(y) = s_{[x]}(y)/s(y).$$

The family of fuzzy sets $\{\mu_{[x]}, [x]\in\Pi\}$ is a fuzzy s-partition of X.

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