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On Some Logical Operators

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Abstract. In this paper an overview on some recent developments concerning the fuzzy logic operators is given. The properties which connect the operators or represent reasoning schemes (in connections with the classical logic) enjoy a special attention. We state some new research ideas in the domain.

Key words: s-norm, t-norm, fuzzy logic, logical operators.

1. Preliminaries

In the classical logical the operators of conjunction (∧), disjunction (∨), negation (–), implication (⇒), equivalence (⇔) are usually given by truth tables, in other words, as applications ∧, ∨, ⇒, ⇔: {0, 1}× {0, 1}→{0, 1}, respectively - : {0, 1}→ {0, 1}. In this frame, for example, $\wedge(x, y)$, usually denoted by $x \wedge y$, is the truth value of a proposition $p \wedge q$, where *p*, *q* are two propositions having the truth value *x*, respectively *y*, where $x, y \in \{0,1\}$. Analogously we can interpret ∨, ⇒, -, ⇔.

It is also well–known that if ∧ (or ∨) and the negation – are given, then ∨ (∧), ⇒, ⇔ can be defined by $x \vee y = \overline{\bar{x} \wedge \bar{y}}$ $\left(x \wedge y = \overline{\bar{x} \vee \bar{y}}\right)$, $x \Rightarrow y = \bar{x} \vee y$, $x \Leftrightarrow y = (x \Rightarrow y) \wedge (y \Rightarrow x)$. We recall the most important tautologies (also the ones given by the above equalities) of the classical logic:

T1) $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

T2) $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

T3) $(p \vee q) \Leftrightarrow (q \vee p)$

T4) $(p \wedge q) \Leftrightarrow (q \wedge p)$

T5) $(p \wedge p) \Leftrightarrow p$

T6) $(p \vee p) \Leftrightarrow p$

T7) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

T8) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

$T_9)$ $(p \wedge q) \Leftrightarrow \overline{\overline{p} \vee \overline{q}}$

$T_{10})$ $(p \vee q) \Leftrightarrow \overline{\overline{p} \wedge \overline{q}}$

$T_{11})$ $p \vee \overline{p}$

$T_{12})$ $\left(p \wedge \overline{p}\right) \Rightarrow q$

$T_{13})$ $(p \Rightarrow q) \Leftrightarrow \left(\overline{p} \vee q\right)$

$T_{14})$ $\overline{p \Rightarrow q} \Leftrightarrow \left(p \wedge \overline{q}\right)$

$T_{15})$ $\overline{p \Leftrightarrow q} \Leftrightarrow \left(p \Leftrightarrow \overline{q}\right)$

$T_{16})$ $\overline{\overline{p}} \Leftrightarrow p$

$T_{17})$ $p \Rightarrow p$

$T_{18})$ $\left[(p \wedge q) \Rightarrow r\right] \Leftrightarrow \left[p \Rightarrow (q \Rightarrow r)\right]$

$T_{19})$ $\left[(p \wedge q) \Rightarrow r\right] \Leftrightarrow \left[\left(p \wedge \overline{r}\right) \Rightarrow \overline{q}\right]$

$T_{20})$ $\left[(p \Rightarrow q) \wedge (q \Rightarrow r)\right] \Rightarrow \left[p \Rightarrow r\right]$

$T_{21})$ $(p \Rightarrow q) \vee (q \Rightarrow p)$

$T_{22})$ $(p \Rightarrow q) \Leftrightarrow \left(\overline{q} \Rightarrow \overline{p}\right)$

$T_{23})$ $(p \Leftrightarrow q) \Leftrightarrow \left(\overline{p} \Leftrightarrow \overline{q}\right)$

$T_{24})$ $(p \Leftrightarrow q) \Leftrightarrow (q \Leftrightarrow p)$

$T_{25})$ $(p \Leftrightarrow q) \Leftrightarrow \left[(p \Rightarrow q) \wedge (q \Rightarrow p)\right]$

$T_{26})$ $\left[(p \Leftrightarrow q) \wedge (q \Leftrightarrow r)\right] \Rightarrow \left[p \Leftrightarrow r\right]$

$T_{27})$ $p \Rightarrow (q \Rightarrow p)$

$T_{28})$ $\left[\overline{p} \Rightarrow \left(q \wedge \overline{q}\right)\right] \Leftrightarrow p$

$T_{29})$ $(p \Rightarrow q) \Rightarrow \left[(q \Rightarrow r) \Rightarrow (p \Rightarrow r)\right]$

$T_{30})$ $\left[(p \Rightarrow r) \wedge (q \Rightarrow r)\right] \Rightarrow \left[(p \vee q) \Rightarrow r\right]$

$T_{31})$ $\left[(p \Rightarrow q) \wedge (p \Rightarrow r)\right] \Rightarrow \left[p \Rightarrow (q \wedge r)\right]$

$T_{32})$ $\left[(p \Rightarrow q) \wedge (r \Rightarrow s)\right] \Rightarrow \left[(p \wedge r) \Rightarrow (q \wedge s)\right]$

$T_{33})$ $\left[(p \Rightarrow q) \vee (r \Rightarrow s)\right] \Rightarrow \left[(p \wedge r) \Rightarrow (q \vee s)\right]$

We notice that:

- the results above are not independent (for example, T_{14} follows from T_9, T_{10} and T_{13});
- using the operator order of priority it is possible to skip some parentheses (doing so the formulas are less suggestive);
- the given list is not exhaustive, for example we can add:

T_{34}) $\left[(p\vee q)\wedge(p\Rightarrow r)\wedge(q\Rightarrow s)\right]\Rightarrow(r\vee s)$

T_{35}) $p\wedge(p\vee q)\Leftrightarrow p$

T_{36}) $p\vee(p\wedge q)\Leftrightarrow p$

T_{37}) $\left[(p\Rightarrow q)\Rightarrow q\right]\Leftrightarrow\left[(q\Rightarrow p)\Rightarrow p\right]$

T_{38}) $\left[(p\Rightarrow p)\Rightarrow p\right]\Leftrightarrow p$

We distinguish three types of assertions:

- properties of operators;
- properties which connect the operators;
- properties which express schemes of reasoning.

In the frame of the interpretation given at the beginning of the paper (x, y, ... $\in\{0,1\}$ are truth values of the propositions p, q, ...) we can rewrite some of the above tautologies. For example we have $(x\wedge y)\wedge r = x\wedge(y\wedge z)$ (the "$\Leftrightarrow$" from T_1 becomes " = "), etc. In the same manner one can rewrite T_2 – T_{10}, T_{13} - T_{16}, T_{18}, T_{19}, T_{22} – T_{25}), etc.

Using this interpretation, one obtains:

T_{39}) $x\wedge 1 = x$;

T_{40}) $x\vee 0 = x$;

T_{41}) $\overline{0}=1$; $\overline{1}=0$;

T_{42}) from $x_1\le x_2$ it follows that $(x_1\wedge y)\le(x_2\wedge y)$;

T_{43}) from $x_1\le x_2$ it follows that $(x_1\vee y)\le(x_2\vee y)$;

T_{44}) from $x\le y$ it follows that $\overline{y}\le\overline{x}$;

T_{45}) from $x_1\le x_2$ it follows that $(x_1\Rightarrow y)\ge(x_2\Rightarrow y)$;

T_{46}) from $y_1\le y_2$ it follows that $(x\Rightarrow y_1)\le(x\Rightarrow y_2)$;

T_{47}) $(0\Rightarrow 0)=(0\Rightarrow 1)=(1\Rightarrow 1)=1,\ (1\Rightarrow 0)=0$;

T_{48}) $x\wedge 0=0$;

T_{49}) $x\vee 1=1$;

T_{50}) $x\wedge\overline{x}=0$;

T_{51}) $x\vee\overline{x}=1$.

From the point of view of the possibility of generalizations we are interested in choosing some basic (in the sense of quasi characteristic) properties. In this context we remember that by **Boolean Algebra** one understands (using the definition introduced by Whitehead) a nonempty set B (containing at least two elements) endowed with the operations $\wedge$, $\vee$: $B \times B \to B$, - : $B \to B$, such that the following conditions are satisfied:

i) $x \vee y = y \vee x$; $x \wedge y = y \wedge x$;

ii) $x \vee (y \vee z) = (x \vee y) \vee z$; $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;

iii) $x \vee (x \wedge y) = x$; $x \wedge (x \vee y) = x$;

iv) $x \vee x = x$; $x \wedge x = x$;

v) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$; $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

vi) there exists $1 = \sup L$, $0 = \inf L$, where sup and inf are relative to the (partial) order given by: $a \le b$ iff $a \wedge b = a$ (or, equivalently $a \vee b = b$);

vii) $x \vee 1 = x$, $x \wedge 1 = x$;

viii) $x \vee \overline{x} = 1$, $x \wedge \overline{x} = 0$.

We remark that:

- the above axioms are not independent (they were presented in the manner above for symmetry reasons);
- the connections with $T_1 - T_8$, T_{39}, T_{40}, T_{49}, T_{50} are immediate.

Other possibilities to define the Boolean Algebras are given by Huntington, Bernstein, etc. In these equivalent ways some other properties of the set $\{T_1, \ldots, T_{50}\}$ are used.

For example, a Boolean Algebra can be defined as a nonempty set *B* (containing at least two elements), together with $\Rightarrow$: $B \times B \to B$ such that:

i) $\exists! 0 \in B : (0 \Rightarrow x) = (x \Rightarrow x), \forall x \in B$;

ii) $[(x \Rightarrow y) \Rightarrow y] = [(y \Rightarrow x) \Rightarrow x]$;

iii) $\{\{[(x \Rightarrow y) \Rightarrow y] \Rightarrow z\} \Rightarrow z\} = \{\{x \Rightarrow [(y \Rightarrow z) \Rightarrow z]\} \Rightarrow [(y \Rightarrow z) \Rightarrow z]\}$

iv) $[(x \Rightarrow x) \Rightarrow x] = x$;

v) $\{\{[(x \Rightarrow (y \Rightarrow 0)) \Rightarrow 0] \Rightarrow [(x \Rightarrow y) \Rightarrow 0]\} \Rightarrow [(x \Rightarrow y) \Rightarrow 0]\} = x$;

vi) $\{[(x \Rightarrow 0) \Rightarrow y] \Rightarrow y\} = (x \Rightarrow y)$.

The other operators are defined in this context by:

$$x \vee y = [(x \Rightarrow y) \Rightarrow y];\ \overline{x} = (x \Rightarrow 0),$$

$$x \wedge y = \{\{[(x \Rightarrow 0) \Rightarrow (y \Rightarrow 0)] \Rightarrow (y \Rightarrow 0)\} \Rightarrow 0\}.$$

A classical example of a Boolean Algebra is $(\{0,1\}, \wedge, \vee, ^-)$.

In the following the truth set {0,1} will be replaced (in order to increase the adequacy to the real world) by an ordered set *L*. Some popular choices are $L=[0,1]$ or $L=\left\{0,\frac{1}{n},\frac{2}{n},...,\frac{n-1}{n},1\right\}, n\in\mathbb{N}, n\geq 2$, with the usual order.

Some useful results in this area are contained in [1], [2], …, [7].

2. The case $L=[0,1]$.

In this case as an operator of **conjunction** one considers a *t-norm*, i.e. a function $t:L\times L\to L$ satisfying the following conditions:

i) $t(x,y)=t(y,x)$;

ii) $t(x,t(y,z))=t(t(x,y),z)$;

iii) from $x_1\leq x_2$ it follows that $t(x_1,y)\leq t(x_2,y)$;

iv) $t(x,1)=x(=t(1,x))$,

for any $x,x_1,x_2,y,z\in L$.

It is clear that, by denoting $t(x,y)=x\wedge y$, we obtain T_1, T_4, T_{39}, T_{42}.

Examples.

i) $t_1:L\times L\to L$, $t_1(x,y)=\max\{x+y-1,0\}$ (bounded difference);

ii) $t_2:L\times L\to L$, $t_2(x,y)=xy$ (algebraic product);

iii) $t_3:L\times L\to L$, $t_3(x,y)=\min\{x,y\}$ (minimum);

iv) $t_h:L\times L\to L$, $t_h(x,y)=\dfrac{xy}{x+y-xy}$ (Hamacher product);

v) $t_l:L\times L\to L$, $t_l(x,y)=\dfrac{xy}{1+(1-x)(1-y)}$ (Einstein product);

vi) $t_w:L\times L\to L$ $t_w=\begin{cases}x, \text{ if } y=1;\\ y, \text{ if } x=1;\\ 0, \text{ otherwise.}\end{cases}$ (drastic product).

As an operator of **disjunction** one takes an *s-norm*, i.e. $s:L\times L\to L$, such that:

i) $s(x,y)=s(y,x)$;

ii) $s(x,s(y,z))=s(s(x,y),z)$;

iii) from $x_1\leq x_2$ it follows that $s(x_1,y)\leq s(x_2,y)$;

iv) $s(x,0)=x(=s(0,x))$,

for any $x,x_1,x_2,y,z\in L$.

Examples.

i) $s_1 : L\times L \to L,\quad s_1(x,y) = \min\{x+y, 1\}$ (bounded sum);

ii) $s_2 : L\times L \to L,\quad s_2(x,y) = x+y-xy$ (algebraic sum);

iii) $s_3 : L\times L \to L,\quad s_3(x,y) = \max\{x,y\}$ (maximum);

iv) $s_h : L\times L \to L,\quad s_h(x,y) = \dfrac{x+y-2xy}{x-xy}$ (Hamacher sum);

v) $s_l : L\times L \to L,\quad s_l(x,y) = \dfrac{x+y}{1+xy}$ (Einstein sum);

vi) $s_w : L\times L \to L\quad s_w = \begin{cases} x, \text{ if } y=0; \\ y, \text{ if } x=0; \\ 1, \text{ otherwise.} \end{cases}$ (drastic sum).

Remark 1. For any $x, y\in L$ we have
$t_w(x,y) \le t_1(x,y) \le t_l(x,y) \le t_2(x,y) \le t_h(x,y) \le t_3(x,y) \le s_3(x,y) \le s_h(x,y) \le$
$\le s_2(x,y) \le s_l(x,y) \le s_1(x,y) \le s_w(x,y)$.

An application $c : L\times L \to L$ satisfying:

i) $c(0)=1, c(1)=0$;

ii) from $x\le y$ it follows that $c(y)\le c(x)$, for any $x, y\in L$;

iii) $c(c(x)) = x$, for any $x\in L$,

is called a **negation**.

Examples.

i) $c' : L\to L,\quad c'(x) = 1-x$ (classical version);

ii) $c'' : L\to L,\quad c''(x) = (1-x^m)^{\frac{1}{m}}, m\in N, m\ge 2$ (version Yager);

iii) $c''' : L\to L,\quad c'''(x) = \dfrac{1-x}{1+\lambda x}, \lambda\in R, \lambda\ne -1$ (version Sugeno).

For a negation *c*, a *t*-norm *t* and an *s*-norm *s* are called *dual to each other with respect to c* if $s(x,y) = c(t(c(x), c(y)))$, or $t(x,y) = c(s(c(x), c(y)))$.

Examples. t_i, s_i, where i = 1, 2, 3; t_h, s_h; t_l, s_l; t_w, s_w are couples of a *t*-norm and an *s*-norm which are dual to each other with respect to *c*'.

An **implication** operator is an application $I : L\times L \to L$ satisfying:

i) from $x_1 \le x_2$ it follows that $I(x_1, y) \ge I(x_2, y)$;

ii) from $y_1 \le y_2$ it follows that $I(x, y_1) \le I(x, y_2)$;

iii) $I(0,0)=I(0,1)=I(1,1)=1, I(1,0)=0$.

Remark 2. If I is an implication, then $I(0,x)=1, I(x,1)=1$ for all $x\in L$.

Examples.

i) $I_1 : L\times L \to L,\quad I_1(x,y)=\min\{1,1-x+y\}$ (version Lukasiewicz);

ii) $I_2 : L\times L \to L,\quad I_2(x,y)=\max\{1-x,\min\{x,y\}\}$ (version Zadeh);

iii) $I_3 : L\times L \to L,\quad I_3(x,y)=\max\{1-x,y\}$ (version Kleene – Dienes);

iv) $I_4 : L\times L \to L,\quad I_4(x,y)=1-x+xy$ (version Reichenbach).

Remark 3. For any $x,y\in L,\quad I_2(x,y)\le I_3(x,y)\le I_4(x,y)\le I_1(x,y)$.

We obtain:

Proposition 1. Let $c:L\to L$ be a negation. Then:

i) c is a continuous function;

ii) from $x>y$ it follows that $c(x)<c(y)$;

iii) there exists a unique $e\in L$ such that $c(e)=e$.

Proposition 2. Let t be a t-norm and let c be a negation. Then:

i) the application given by $s(x,y)=c\big(t(c(x),c(y))\big)$ is an s-norm;

ii) the application given by $I(x,y)=s(c(x),y)$ is an implication;

iii) the implication I defined above is such that:

a) $I(x,y)=I(c(y),c(x))$;

b) $I(x,y)\ge \max\{c(x),y\}$;

c) $I(1,x)=x$;

d) $I(x,I(y,z))=I(y,I(x,z))$.

Examples. For $t=t_i, c=c'$ we have $s=s_i, I=I_i, i=1,2,3$.

Remark 4. One can obtain a similar result starting with an s-norm and a negation.

Remark 5. Another possibility to obtain an implication (having a t-norm t) is given by $J:L\times L\to L,\quad J(x,y)=\sup\{z \mid t(x,z)\le y\}$.

In this context we have:

Proposition 3. If $x\le y$, then $J(x,y)=1$.

Examples. i) For t_1 we obtain $J(x,y)=\begin{cases}1, & x\le y\\ 1-x+y, & x>y\end{cases} = I_3$;

ii) For t_2 we obtain $J\left(x,y\right)=\begin{cases}1, & x\le y\\ \dfrac{y}{x}, & x>y\end{cases}$;

iii) For t_3 we obtain $J\left(x,y\right)=\begin{cases}1, & x\le y\\ y, & x>y\end{cases}$;

Remark 6. For a *t*-norm *t*, an *s*-norm *s*, and a negation *c* one can define $I:L\times L\to L, I(x,y)=s(c(x),t(x,y))$. But *I* is a implication only under strong conditions on *t* and *s*.

Now we suppose that the $I:L\times L\to L$ is an implication and consider $c:L\to L$, $s:L\times L\to L$, $t:L\times L\to L$ given by

$c\left(x\right)=I\left(x,0\right)$; $s\left(x,y\right)=I\left(c\left(x\right),y\right)$; $t\left(x,y\right)=c\left(s\left(c\left(x\right),c\left(y\right)\right)\right)$,

for all $x,y\in L$.

Proposition 4. i*)* If $I\left(x,y\right)=I\left(c\left(y\right),c\left(x\right)\right)$, for all $x,y\in L$ then *c* is a negation if and only if $I\left(1,x\right)=x$ for all $x\in L$;

ii) If $I\left(x,y\right)=I\left(c\left(y\right),c\left(x\right)\right)$, $I\left(x,I\left(y,z\right)\right)=I\left(y,I\left(x,z\right)\right)$ and $I\left(1,x\right)=x$, then *s* is an *s*-norm (and t is its dual t-norm). We have also *s*(*c*(*x*), *y*) = *I*(*x*, *y*).

We can reformulate Proposition 2.iii) and Proposition 4.ii), obtaining:

Proposition 5. Let $s,I:L\times L\to L$ such that $s\left(x,y\right)=I\left(c\left(x\right),y\right)$ (or, equivalently, $I\left(x,y\right)=s\left(c\left(x\right),y\right)$, where c is a negation on L. Then s is an s-norm if and only if I is an implication satisfying a), c), d) from Proposition 2.

Proposition 6. Let $t,I:L\times L\to L$ such that $I\left(x,y\right)=\sup\left\{z\middle|t\left(x,z\right)\le y\right\}$ and c be a negation on L. Then t is a t-norm if and only if I is an implication satisfying $I\left(1,x\right)=x$, $I\left(x,x\right)=1$ for all $x\in L$, and $I\left(x,y\right)=I\left(c\left(y\right),c\left(x\right)\right)$ for all $x,y\in L$.

Another way to obtain s (and t), c and I is given by choosing an application $\varphi:L\to L$ satisfying the conditions:

i) $\varphi\left(0\right)=0,\varphi\left(1\right)=1$;

ii) φ is continuous;

iii) φ is strictly increasing.

In this context we put:

$c\left(x\right)=\varphi^{-1}\left(1-\varphi\left(x\right)\right)$;

$s\left(x,y\right)=\varphi^{-1}\left(\min\left\{\varphi\left(x\right)+\varphi\left(y\right),1\right\}\right)$;

$t(x,y)=\varphi^{-1}\left(\max\{\varphi(x)+\varphi(y)-1,0\}\right)$;

$I(x,y)=\varphi^{-1}\left(\min\{1-\varphi(x)+\varphi(y),1\}\right)$.

We have:

Proposition 7. *i) c* is a negation;

ii) s is an s-norm (satisfying $s(x,c(x))=1$);

iii) t is a t-norm (satisfying $t(x,c(x))=0$);

iv) s and *t* as above are dual to each other;

v) I is an implication;

vi) $c(x)=I(x,0)$.

We can define also:

$c(x)=\varphi^{-1}(1-\varphi(x))$;

$s(x,y)=\varphi^{-1}(\varphi(x)+\varphi(y)-\varphi(x)\cdot\varphi(y))$;

$t(x,y)=\varphi^{-1}(\varphi(x)\cdot\varphi(y))$;

$I(x,y)=\varphi^{-1}(1-\varphi(x)+\varphi(x)\cdot\varphi(y))$.

Proposition 8. i) *s*, *t* are an *s*-norm and, respectively, a *t*-norm, dual to each other;

ii) *I* is an implication.

Examples. As an application φ we can take for instance:

i) $\varphi: L\to L, \quad \varphi(x)=x$;

ii) $\varphi: L\to L, \quad \varphi(x)=x^m, m\in N, m>1$.

3. The case $L=\left\{0,\frac{1}{n},\frac{2}{n},...,\frac{n-1}{n},1\right\}$.

In this case we keep the same definitions for a *t*-norm, *s*-norm, negation, implication. We can transfer as examples: t_1, t_3, s_1, s_3, c', I_1, I_2, I_3. The constructions $I(x,y)=s(c(x),y)$; $J(x,y)=\sup\{z \mid t(x,z)\le y\}$ are also used. We remark that $c'(x)=1-x$ gives $c'\left(\frac{i}{n}\right)=\frac{n-i}{n}, i=0,...,n$.

Proposition 9. The above defined negation is the unique negation which can be defined on *L*.

Remark 7. The propositions 5 and 6 can be rewritten for *L* as above.

We have also:

Proposition 10. *i)* (L,t_2,s_2,c') is a De Morgan algebra;

ii) (L, t_2, s_2, I_2) is a Heyting algebra.

4. Concluding remarks

The study will be continued for finding other dependencies among the possible properties of logical operators. Another open problem is the analysis of the possibility of transfer of the properties of s or t to I. Last but not least we remark the necessity to study the organization of L as an algebra (Boole, De Morgan, Heyting).

Finally we propose a new way to obtain logical operators: for an application $\psi : L \times L \to L$ we consider $c: L \to L$, $c(x) = \psi(x, x)$, $t, s, I : L \times L \to L$,

$$t(x,y) = \psi(c(x), c(y)),\ s(x,y) = c(\psi(x,y)),\ I(x,y) = c(\psi(c(x), y)).$$

From the conditions (on ψ) such that c, t, s, I are logical operators one obtains the (Bernstein type) conditions:

$$\psi(\psi(y,x), \psi(c(y),x)) = x; \quad \psi(c(x), \psi(c(y),z)) = c(\psi(\psi(y, c(x), \psi(c(z), c(x)))))$$

where $c(\alpha) = \psi(\alpha, \alpha)$.

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