F.S.A.I., Vol. 10, No. 3, pp. 5-13, 2004

Fuzzy Games and Coherent Fuzzy Previsions

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Abstract. We consider some possible extensions of the coherent prevision of random numbers, introducing suitable definitions on prevision of "random fuzzy numbers". We consider also some applications to the "fuzzy games".

Key words: Coherent previsions, Fuzzy games, Fuzzy coherent previsions

1. Introduction

In the classical game theory, in a game with a set of *nature states*, as *profit* of a strategy is considered the *prevision* of the *random profit* that depends on the nature state that happens.

Such prevision can be calculated if we know the probabilities of all the nature states. An alternative, proposed by de Finetti, [5], and gone further in details in following papers (see, e. g. [3], [12]), is to assign, in a subjective way, a *coherent prevision* to a given set of random numbers containing the *random profit*.

In particular the *events* are random numbers with codomain {0,1} and the nature states are the *atoms* with respect to the events.

In this paper we consider particular *fuzzy games* and we extend the concept of *coherent prevision* to the *random fuzzy numbers*.

2. Random fuzzy numbers and fuzzy games

Let R be the set of real numbers.

We recall some definitions on the fuzzy numbers.

Definition 2.1. A fuzzy number (FN) is a function $\varphi : R \to [0,1]$ such that:

1) *(FN1)* $\exists a,b \in R, a \le b$, *such that* $\varphi(x) = 0$ *in* $R - [a,b]$ *and* $\varphi(x) > 0$ *in* (a,b);

2) *(FN2)* $\exists c,c' \in R, c \le c'$, *such that* $\varphi(x) = 1$, $\forall x \in [c,c']$;

3) (FN3) φ is increasing for $x \le c$ and decreasing for $x \ge c'$.

The interval $[a,b]$ is the *support* of φ, noted $S(\varphi)$. The elements a and b are, respectively, the *infimum* and the *supremum* of φ. The set $[c,c']$ is the *core* of φ, noted $C(\varphi)$. We write $\varphi = [a,c,c',b]$ to indicate that φ is a FN with support $[a,b]$ and core $[c = c']$.

In this paper a fuzzy number $\varphi = [a,c,c',b]$ is said to be:

continuous if φ is a function continuous in $R - \{a,b\}$;

simple if $[c,c']$;

degenerate if $a = c = c' = b$;

linear if the restrictions of φ to the intervals $[a,c]$ and $[c',b]$ are linear functions.

We denote with:

Φ the set of all the fuzzy numbers;

$\Im$ the set of the continuous ones;

L the set of the linear fuzzy numbers.

An important subset of **L** is the set **T** of *trapezoidal fuzzy numbers* (TFN).

A TFN $\varphi = [a,c,c',b]$ is a fuzzy number such that:

4) $$a \le x < c \Rightarrow \varphi(x) = (x-a)/(c-a), c' < x \le b \Rightarrow \Rightarrow \varphi(x) = (b-x)/(b-c'). \quad (1)$$

A simple TFN $\varphi = [a,c,c',b]$, with $c = c'$, is called *triangular fuzzy number* (TRFN) and is noted $\varphi = (a, c, b)$. Let Δ be the set of TRFN's.

The set R of real numbers is identified with the set of degenerate fuzzy numbers, by considering any real number c equal to the fuzzy number with support {c}. Then we have $R \subseteq \Delta \subseteq T \subseteq L \subseteq \Im \subseteq \Phi$.

We recall that any operation σ on R can be extend to the set S of all the fuzzy sets with universe R. Precisely, if φ and γ are fuzzy sets with universe R, we put:

5) $$\varphi\sigma y : z \in R \to \sup_{x\sigma y = z} (\min(\varphi(x), \gamma(y))). \quad (2)$$

If H and K are subsets of S such that $R \subseteq H \subseteq K$ and $(\varphi \in H, y \in K) \Rightarrow \varphi\sigma y \in K$ we say that σ is *extensible* to $H \times K$. In particular, if

$H = K$ we say that σ is *extensible* to K and if $H = R$ we say that σ is *extensible* as *external operation* on K with R as set of scalars.

If K is a set of fuzzy numbers, with $R \subseteq K$, we denote with K^* the set of elements $\varphi = \left[a_\varphi, c_\varphi, c'_\varphi, b_\varphi \right]$ of K such that $0 \notin \left(a_\varphi, b_\varphi \right)$. We say that an operation σ on R is *quasi extensible* to K if it is extensible to $K^* \times K$.

Since $R \subseteq K^*$, if σ is *quasi extensible* to K, σ is also *extensible* as *external operation* on K with R as set of scalars.

For example, if $K \in \{\Phi, \Im, L, T, \Lambda\}$, the addition + is extensible on K and the multiplication · is quasi extensible on K.

Definition 2.2. We say that a subset K of Φ is *regular* if:

6) (R1) contains R;

7) (R2) + is extensible to K, and · is quasi extensible to K.

We say that $K \subseteq \Phi$ is *weakly regular* if we have (R1) and:

8) (R2W) + is extensible to K, and · is extensible to $R \times K$

To extend game and decision theories to the general case in which utilities are fuzzy numbers it is necessary to introduce pre-order relations on Φ or, at least, on some of its weakly regular subsets

There are many possible pre-orders. In this paper we assume as *first pre-order relation* the following:

Definition 2.3. Let $\varphi = \left[a_\varphi, c_\varphi, c'_\varphi, b_\varphi \right]$ and $y = \left[a_y, c_y, c'_y, b_y \right]$ be two elements of Φ. We write $\varphi \leq_1 \gamma$ if we have:

9) $c_\varphi \leq c_y$ and $\forall x \in R, x \leq c_\varphi \Rightarrow y(x) \leq \varphi(x)$ and $x \geq c_y \Rightarrow y(x) \geq \varphi(x)$ *(3)*

We can prove that (3) implies also $c'_\varphi \leq c'_y$ and that $\leq_1$ is a *partial order* relation. We can extend $\leq_1$ to a more wide subset of Φ^2 in many ways, but in general we can obtain only a *partial pre-order* relation. Nevertheless, if we consider a suitable weakly regular subset K of Φ we can obtain a *total pre-order* relation on K. This is sufficient for game and decision theories.

Some of the most common of such extensions are:

(a) Relation ("*trapezoidal*") $\leq_2$ defined as:

10) $\forall\varphi=\left[a_{\varphi},c_{\varphi},c_{\varphi}^{'},b_{\varphi}\right]$ and $y=\left[a_{y},c_{y},c_{y}^{'},b_{y}\right]\in\Phi,$
$\varphi\leq_{2} y\Leftrightarrow a_{\varphi}\leq a_{y},c_{\varphi}\leq c_{y},c_{\varphi}^{'}\leq c_{y}^{'},b_{\varphi}\leq b_{y.}$ *(4)*

The $\leq_2$ is a *partial pre-order*, extension of $\leq_1$, on Φ. The restriction of $\leq_2$ to the regular subset of trapezoidal fuzzy numbers if a *partial order* relation, coincident with $\leq_1$.

(b) Relation ("*crisp*") $\leq_3$ defined as:

11) $\forall\varphi=\left[a_{\varphi},c_{\varphi},c_{\varphi}^{'},b_{\varphi}\right]$ and $y=\left[a_{y},c_{y},c_{y}^{'},b_{y}\right]\in\Phi,$
$\varphi\leq_{3} y\Leftrightarrow c_{\varphi}\leq c_{y},c_{\varphi}^{'}\leq c_{y}^{'}.$ *(5)*

The $\leq_3$ is also a *partial pre-order*, extension of $\leq_2$, on Φ. The restriction of $\leq_3$ to the simple fuzzy numbers if a *total pre-order* relation.

Definition 2.4. Let K be a subset of Φ and let Π be a partition of the certain event Ω. A random fuzzy number with domain Π and codomain K is a function

$X: E\in\Pi\rightarrow X(E)\in K.$

We put:

$K'=X(\Pi);$

for any $z\in K', X^{-1}(z)=\cup\left\{E\in\Pi: X(E)=z\right\},$

$\Pi^{*}=\left\{X^{-1}(z), z\in K'\right\};$

for any $z\in K', A\in\Pi^{*}, X^{*}(A)=z\Leftrightarrow A=X^{-1}(z).$

The *reduced form* of X is the random fuzzy number

$X^{*}: A\in\Pi^{*}\rightarrow X^{*}(A)\in K.$

There are many possible definitions of fuzzy games as generalization of the classical games. In this paper we assume the following:

Definition 2.5. A *fuzzy game* for n persons is a complex $\Gamma=\left(\Sigma_1,\Sigma_2,...,\Sigma_n,\Theta,\varphi_1,\varphi_2,...,\varphi_n,K\right)$ where

$\Sigma_1=\left\{\sigma_1^1,\sigma_2^1,...,\sigma_{m1}^1\right\}$, $\Sigma_2=\left\{\sigma_1^2,\sigma_2^2,...,\sigma_{m2}^2\right\}$, …, $\Sigma_n=\left\{\sigma_1^n,\sigma_2^n,...,\sigma_{mn}^n\right\}$,

are the sets of "*pure strategies*", respectively, for the players 1, 2, …, n;

$\Theta=\left\{\theta_1,\theta_2,...,\theta_n\right\}$ is a partition of the certain event, called "the set of *nature states*";

K is a weakly regular subset of Φ;

$\varphi_i : \Sigma_1 \times \Sigma_2 \times ... \times \Sigma_n \times \Theta \to K, i = 1, 2, ..., n$ are applications, called *fuzzy random profit functions*.

The fuzzy number $\varphi_i\left(\sigma_{r1}^1, \sigma_{r2}^2, ..., \sigma_{rn}^n, \theta_u\right)$ means "the fuzzy profit" by the player *i*, in the hypothesis that are chosen the pure strategies $\sigma_{r1}^1, \sigma_{r2}^2, ..., \sigma_{rn}^n$ and happens θ_u.

The fuzzy game is said to be:

deterministic if $\Theta = \{\Omega\}$;

crisp game if K = R.

If Γ is a not deterministic fuzzy game, for every $(i, r1, r2, ..., rn) \in \{1, 2, ..., n\} \times \{1, 2, ...m1\} \times ... \{1, 2, ..., mn\}$ we can consider the function:

12) $\delta_{i,r1,r2,...,rn} : \theta \in \Theta \to \varphi_i\left(\sigma_{r1}^1, \sigma_{r2}^2, ..., \sigma_{rn}^n, \theta\right) \in K$. *(6)*

This is a random fuzzy number and it is reduced to a *de Finetti* random number if *K* = *R*.

For the crisp games the random fuzzy number $\delta_{i,r1,r2,...,rn}$ is usually considered *equivalent* to its prevision $P\left(\delta_{i,r1,r2,...,rn}\right)$, where, following [5] (see also [3], [12]), we assess a coherent prevision on a set of random numbers containing the functions (6).

From this point of view a random crisp game Γ is replaced with a deterministic crisp game Γ*, considered *equivalent* to Γ.

In this paper we intend generalize this procedure to the fuzzy games, by introducing suitable definitions of *fuzzy prevision*.

3. Crisp previsions and fuzzy previsions

From now on we consider only bounded random numbers. We remember some fundamental definitions and theorems on coherent previsions (see [5], [3], and [12]).

Definition 3.1. Let *S* be a non empty set of bounded random numbers.

A *prevision* on *S* is a function *P*: *S* → *R* such that:

13) (P1) $\forall a, b \in R, \forall X \in S, a \le X \le b \Rightarrow a \le P(X) \le b$ *(mean property);*

14) (P2) $\forall X, Y \in S, (X + Y \in S) \Rightarrow P(X + Y) = P(X) + P(Y)$ *(additivity).*

We say that the prevision P is *coherent* if there exists an extension of P to the vector space $V(S)$ generated by S, that is to the set of linear combinations of elements of S.

Let S be a non empty set of bounded random numbers and let P be a prevision on S. Some important consequences of the previous definition are:

Proposition 3.1. *We have*

15) (C1) (positivity) $\forall X \in S, X \geq 0 \Rightarrow P(X) \geq 0$;

16) (C2) (monotonicity) $\forall X, Y \in S, (Y - X \in S, X \leq Y) \Rightarrow P(X) \leq P(Y)$.

Proposition 3.2. *If P is coherent then there is only an extension P* of P to V(S) and we have:*

17) (P3) $$\begin{aligned}&\forall n \in N, \forall X_1, X_2, ..., X_n \in S, \forall c_1, c_2, ..., c_n \in R,\\ &P^*(c_1 X_1 + c_2 X_2 + ... + c_n X_n) = c_1 P(X_1) + c_2 P(X_2) + ... + c_n P(X_n)\end{aligned}$$

Proposition 3.3. *If P is coherent and P* is its extension to V(S) then:* $\forall n \in N, \forall X_1, X_2, ..., X_n \in V(S), \forall c, c_1, c_2, ..., c_n \in R$, *we have:*

18) (P4) $c_1 X_1 + c_2 X_2 + ... + c_n X_n \leq c \Rightarrow c_1 P^*(X_1) + ... + c_n P^*(X_n) \leq c$;

19) (P5) $c_1 X_1 + c_2 X_2 + ... + c_n X_n \geq c \Rightarrow c_1 P^*(X_1) + ... + c_n P^*(X_n) \geq c$; ;

20) (P6) $c_1 X_1 + c_2 X_2 + ... + c_n X_n = c \Rightarrow c_1 P^*(X_1) + ... + c_n P^*(X_n) = c$; .

Proposition 3.4. *Let S be a non empty set of random numbers and let P be a function defined on S and with values on R.*

If, $\forall n \in N, \forall X_1, X_2, ..., X_n \in S, \forall c, c_1, c_2, ..., c_n \in R$, *(P4) or (P5) holds, then*

P: V(S) $\to$ R given by (P3) is a prevision on V(S) and so P is a coherent prevision on S.*

Now we consider some extensions of previous concepts, to the random fuzzy numbers. We assume K is a subset of Φ containing R and Q_K is the set of fuzzy random numbers with codomain K.

If S is a non empty subset of Q_K and K is weakly regular we denote with $V(S)$ the set of linear combinations of elements of S with coefficients on R, that is the set:

$$V(S) = \{c_1 X_1 + c_2 X_2 + ... + c_n X_n, \text{ with } X_1, X_2, ..., X_n \in S, c_1, c_2, ..., c_n \in R\}.$$

If K is regular it is meaningful also to consider also the set

$$V_K(S) = \{c_1X_1 + c_2X_2 + ... + c_nX_n, \text{ with } X_1, X_2, ..., X_n \in S, c_1, c_2, ..., c_n \in K^*\},$$

whose elements are the linear combinations of elements of *S* with coefficients on *K**.

We propose some possible extensions of the definition 3.1 to the fuzzy random numbers. We consider two classes of extensions: the *fuzzy previsions* and the *crisp previsions*.

Definition 3.2. Let *S* be a non empty set of fuzzy random numbers and let *H* and *K* be two subsets of Φ such that:

$R \subseteq H \subseteq K$;

+ is extensible to *K*;

· is extensible to $H \times K$.

A *fuzzy prevision P* on *S* with respect to ($H, K, \leq_i$), $i \in \{1, 2, 3\}$, is a function $P: S \to K$ such that:

21) (FP1) $\forall a, b \in K, \forall X \in S, a \leq_i X \leq_i b \Rightarrow a \leq_i P(X) \leq_i b$ *(monotonicity);*

22) (FP2) $\forall X, Y \in S, (X + Y \in S) \Rightarrow P(X + Y) = P(X) + P(Y)$ *(additivity);*

23) (FP3) $\forall a \in H, \forall X \in S, (aX \in S) \Rightarrow P(aX) = aP(X)$ *(homogeneity).*

We say that the fuzzy prevision *P* is *coherent* if there exists an extension *P** of *P* to the set $V_H(S)$ of the linear combinations of elements of *S* with coefficient on *H**.

We say also that *P* is a:

- regular fuzzy prevision on K if H = K;
- weakly regular fuzzy prevision on K if H = R.

The previous definition has many particular cases. In particular:

For *R* = *H* = *K* we have the usual definition 3.1.

If *S* is a set of events, that is random numbers with codomain {0, 1}, we have the definition of coherent probability.

If *S* is a set of random numbers that assume values on [0, 1] we can introduce the concept of coherent probability of fuzzy events.

By generalizing the theorems from [5], [12] and [14] we can extend propositions 3.2, 3.3, 3.4 to the fuzzy previsions. Precisely, if we assume that *S* is a non empty set of fuzzy random numbers and *H* and *K* are two subsets of Φ such that $R \subseteq H \subseteq K, +$ is extensible to *K* and · is extensible to $H \times K$, we can prove the following theorems:

Theorem 3.1. *If P is a coherent fuzzy prevision with respect to (H, K,* $\leq_i$*) then there is only an extension P* of P to* $V_H(S)$ *and we have:*

24) (FPE) $\forall n \in N, \forall X_1, X_2, ..., X_n \in S, \forall c_1, c_2, ..., c_n \in H$,

$$P^*\left(c_1X_1 + c_2X_2 + ... + c_nX_n\right) = c_1P\left(X_1\right) + c_2P\left(X_2\right) + ... + c_nP\left(X_n\right).$$

Theorem 3.2. *If P is a coherent fuzzy prevision with respect to (H, K, $\leq_i$), i=1, 2, 3, … and P* is its extension to $V_H(S)$ then*:

$\forall n \in N, \forall X_1, X_2, ..., X_n \in V_H\left(S\right), \forall c_1, c_2, ..., c_n \in H, \forall c \in K$, we have:

25) (FP4) $c_1X_1 + c_2X_2 + ... + c_nX_n \leq_i c \Rightarrow c_1P^*\left(X_1\right) + ... + c_nP^*\left(X_n\right) \leq_i c$ *;*

26) (FP5) $c_1X_1 + c_2X_2 + ... + c_nX_n \geq_i c \Rightarrow c_1P^*\left(X_1\right) + ... + c_nP^*\left(X_n\right) \geq_i c$ *;*

27) (FP6) $c_1X_1 + c_2X_2 + ... + c_nX_n = c \Rightarrow c_1P^*\left(X_1\right) + ... + c_nP^*\left(X_n\right) = c$.

Theorem 3.3. *Let S be a non empty set of fuzzy random numbers and let P be a function defined on S and with values on K.*

If, $\forall n \in N, \forall X_1, X_2, ..., X_n \in S, \forall c \in K, \forall c_1, c_2, ..., c_n \in H$, *(FP4) or (FP5) holds then* $P^* : V_H\left(S\right) \rightarrow K$ *given by (FPE) is a fuzzy prevision on $V_H(S)$ and so P is a coherent prevision on S.*

As alternative to the concept of *fuzzy prevision* we introduce the one of *crisp prevision*. This one is a generalization of the conditions of normalization utilized to treat with fuzzy partitions. (see e. g. [16]).

Definition 3.3. Let *S* be a non empty set of fuzzy random numbers and let *K* be a weakly regular subset of Φ.

A *crisp prevision P* on *S* with regard to $\leq_i, i \in \{1,2,3\}$, is a function *P*: *S*→*R* such that:

28) (CP1) $\forall a, b \in K, \forall X \in S, a \leq_i X \leq_i b \Rightarrow C\left(a\right) \leq P\left(X\right) \leq C\left(b\right)$ *(monotonicity);*

29) (CP2) $\forall X, Y \in S, \left(X + Y \in S\right) \Rightarrow P\left(X + Y\right) = P\left(X\right) + P\left(Y\right)$ *(additivity);*

30) (CP3) $\forall a \in R, \forall X \in S, \left(aX \in S\right) \Rightarrow P\left(aX\right) = aP\left(X\right)$ *(homogeneity).*

We say that the crisp prevision *P* is *coherent* if there exists an extension *P** of *P* to the set *V*(*S*) of the linear combinations of elements of *S* with coefficients on *R*.

The crisp previsions consider only the core of fuzzy numbers. It is particular meaningful if *i* = 3 and we consider simple fuzzy numbers. In this case $\leq_i$ is a total pre-order.

Practically, to obtain the best results, it is convenient to assume K is the set of trapezoidal fuzzy numbers, or in general, if we have to consider the possibility of "a cut" of the support, the set of linear fuzzy numbers.

If we wish to have only an element on the core it is convenient to consider K as the set of simple linear fuzzy numbers.

References

[1] BERTI P., RAGAZZINI E., RIGO P., *Coherent prevision of random elements*, CNR, Istituto per le applicazioni della matematica, Milano (1994).

[2] COLETTI G., SCOZZAFAVA R., *Probabilistic logic in a coherent setting*, Kluwer Academic Publishers, London (2002).

[3] CRISMA L., GIGANTE P., *A notion of coherent conditional prevision for arbitrary random quantities*, in Statistical Methods and Applications, 10, 29-40. Journal of the Italian Statistical Society, vol 3, n 3, 233-243, (2001).

[4] CRISMA L., GIGANTE P., MILLOSSOVICH P., *A notion of coherent prevision for arbitrary random quantities*, Journal of the Italian Statistical Society, vol 3, n 3, 233-243, (1997).

[5] de FINETTI B., *Teoria delle Probabilità*, vol. 1 and 2, Einaudi, Torino (1970).

[6] DUBINS L. E., *Finitely additive conditional probabilities, conglomerability and disintegrations*, The Annals of Probability, 3, 89-99, (1975).

[7] FADINI A., *Introduzione alla teoria degli insiemi sfocati,* Liguori, Napoli (1979).

[8] FERRI B, MATURO A, *An application of the fuzzy set theory to evaluation of urban project,* in New Trends in Fuzzy Systems, Word Scientific, pp. 82-91 (1998).

[9] HOLZER S., *On coherence and conditional prevision*, Boll. UMI, Serie VI, Vol IV, 441-460 (1985).

[10] KAUFMANN, A., *Theory of fuzzy subsets*, Academic Press, New York (1975).

[11] KLIR G.J., YUAN B., *Fuzzy sets and fuzzy logic,* Prentice Hall, N. Jersey (1995).

[12] MATURO A., *Sull'assiomatica di Bruno de Finetti per la previsione e la probabilità coerenti: analisi critica e nuove prospettive*, Periodico di Matematiche, Serie VIII, Vol 3, n. 1, 41-54 (2003).

[13] MATURO A., *Fuzzy events and their probability assessments*, Journal of Discrete Mathematical Sciences & Cryptography, Vol. 3, Nos 1-3, 83-94 (2000).

[14] MATURO A., Grandezze aleatorie fuzzy e loro previsioni per le decisioni in condizione di informazione parziale, in Current Topics in Computer Science, Panfilus, Iasi (Romania) (2004).

[15] MATURO A., TOFAN I., Iperstrutture, strutture fuzzy ed applicazioni, Dierre edizioni, San Salvo (Ch) (2001).

[16] FRANCHINO R., MATURO A., VENTRE A. , VIOLANO A., *Strategie, Processi e Modelli Decisionali per la Gestione dell'Ambiente*, Edizioni Goliardiche, Trieste (2004).

[17] SQUILLANTE M., VENTRE A., *Consistency for uncertainty measure*, International Journal of Intelligent Systems, vol 13, 419-430 (1998).

[18] WALLEY P., *Statistical Reasoning with Imprecise Probabilities*, Chapman and Hall, London (1991).

[19] ZADEH L., *Fuzzy sets,* in Information and Control, vol.8, pp. 338-353 (1965).