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Decomposition of Fuzzy Relations Based on Multi-Resolution Scheme and Its Application to Image Analysis

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Abstract: As an image component analysis method by fuzzy relational structure, a decomposition of a fuzzy relation based on multi-resolution schemes is proposed. The proposed method regards the component analysis as the optimization problem to minimize the cost function of the difference between the image and component fuzzy sets, where the target image is obtained by multi-resolution scheme as image compression method based on fuzzy relational equations. The proposed component analysis method can reduce the computation time and produce the perspective feature of the original image due to solving the decomposition problem of the compressed image. Furthermore, a component analysis of restricted image region can be performed by the proposed method. Through an experiment using the test images extracted from Standard Image DataBAse (SIDBA), it is confirmed that the computation time of the proposed method is decreased into 31.4% of the conventional one, under the condition that the size of original image and compressed one are 256 x 256 and 128 x 128, respectively.

1 Introduction

By using fuzzy relational calculus [1][2], many image processing techniques have been developed [3][5][6][7][8], especially interest of decomposition of a fuzzy relation is still growing as a component analysis of the image [10][11]. The decomposition of fuzzy relation corresponds to an optimization problem to minimize a cost function measuring the difference between the original image and the component fuzzy sets. In order to reduce the computation time, a decomposition method of fuzzy relation based on multi resolution scheme is proposed. The proposed method treats with the target image as the compressed image that is given by a compression and reconstruction method by fuzzy relational equations (ICF) [3][5]. By solving the decomposition problem of the compressed image instead of the original one, the proposed method can obtain the global feature of the original image as the component fuzzy sets and reduce the computation time. Furthermore, the ICF can produce the image compression restricted to an image region, and the proposed method can perform the component analysis of a restricted image region by using the ICF's capability. In an experiment using the test images extracted from Standard Image DataBAse (SIDBA), it is confirmed that the computation time of the proposed method is decreased into 31.4% of the conventional one, under the condition that the size of original image and compressed image is 256 x 256 and 128 x 128, respectively.

2 Decomposition of Fuzzy Relation with Max-Min Composition

An original image of size $M \times N$ (pixels) can be treated as a fuzzy relation $R \in F(\mathbf{X} \times \mathbf{Y})$, $\mathbf{X} = \{x_1, x_2, \ldots, x_M\}$, $\mathbf{Y} = \{y_1, y_2, \ldots, y_N\}$, by normalizing the intensity range of each pixel into $[0,1]$. The decomposition of R consists in the determination of two pairs of fuzzy sets, $\{A_i \in F(\mathbf{X}) \mid i = 1, \ldots, c\}$ and $\{B_i \in F(\mathbf{Y}) \mid i = 1, \ldots, c\}$ such that

$$R(x, y) \approx \tilde{R}(x, y) = \bigvee_{i=1}^{c} \left(A_i(x) \wedge B_i(y)\right), \tag{1}$$

for all $x \in \mathbf{X}$, $y \in \mathbf{Y}$, where c denotes the Schein rank [4] of the fuzzy relation R, that is the smallest integer satisfying Eq. (1). The operators "$\vee$" and "$\wedge$" stand for "max" and "min", respectively. Obviously c measures the approximation performance $\tilde{R}$ of R, in the sense that the possible larger the value c, the better approximation $\tilde{R}$ is. The decomposition problem (Eq. (1)) can be seen as an optimization that minimizes the cost function:

$$\tilde{Q} = \sum_{(x,y) \in \mathbf{X} \times \mathbf{Y}} \left(R(x, y) - \bigvee_{i=1}^{c} \left(A_i(x) \wedge B_i(y)\right) \right)^2. \tag{2}$$

A solution based on the gradient method has been presented in [11]. Since this method is iterative, we denote by ``iter'' the iteration step. Thus the detailed notation of Eq. (2) is given by

$$\tilde{Q}_{iter} = \sum_{(x,y) \in \mathbf{X} \times \mathbf{Y}} \left(R(x, y) - \bigvee_{i=1}^{c} \left(A_i^{(iter)}(x) \wedge B_i^{(iter)}(y)\right) \right)^2, \tag{3}$$

is the mentioned fuzzy sets iteratively are defined by

$$A_i^{(iter+1)}(x) = A_i^{(iter)}(x) - \alpha \frac{\partial \tilde{Q}_{iter}}{\partial A_i^{(iter)}(x)}, \tag{4}$$

$$B_i^{(iter+1)}(y) = B_i^{(iter)}(y) - \alpha \frac{\partial \tilde{Q}_{iter}}{\partial B_i^{(iter)}(y)}, \tag{5}$$

for all $x \in \mathbf{X}$, $y \in \mathbf{Y}$, where α is a learning rate. For simplicity of notation, by setting $\tilde{Q} = \tilde{Q}_{iter}$, $A_i = A_i^{(iter+1)}$, and $B_i = B_i^{(iter+1)}$, the derivation of the cost function $\tilde{Q}$ with respect to $A_l(x')$ can be written for $l = 1, 2, \ldots, c$ as

$$\frac{\partial \tilde{Q}}{\partial A_l(x')} = \frac{\partial}{\partial A_l(x')} \sum_{(x,y)\in \mathbf{X}\times\mathbf{Y}} \left(R(x,y) - \bigvee_{i=1}^{c} \left(A_i(x) \wedge B_i(y) \right) \right)^2$$

$$= -2 \sum_{y\in\mathbf{Y}} \left(R(x',y) - \bigvee_{i=1}^{c} \left(A_i(x') \wedge B_i(y) \right) \right) \cdot \frac{\partial}{\partial A_l(x')} \left(\bigvee_{i=1}^{c} \left(A_i(x') \wedge B_i(y) \right) \right)$$

$$= -2 \sum_{y\in\mathbf{Y}} \left(R(x',y) - \bigvee_{i=1}^{c} \left(A_i(x') \wedge B_i(y) \right) \right) \cdot \varphi\left(A_l(x') \wedge B_l(y), \bigvee_{i=1, i\neq l}^{c} \left(A_i(x') \wedge B_i(y) \right) \right) \cdot \psi\left(A_l(x'), B_l(y) \right) \tag{6}$$

being the functions φ and ψ defined by

$$\varphi(a,b) = \begin{cases} 1 & \text{if} \quad a \geq b, \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

$$\psi(a,b) = \begin{cases} 1 & \text{if} \quad a \leq b, \\ 0 & \text{otherwise}. \end{cases} \tag{8}$$

The above process is performed until a final criterion, e.g., if

$$\tilde{Q}_{iter+1} - \tilde{Q}_{iter} < \varepsilon \tag{9}$$

is satisfied, where ε is a required threshold. For the computation of the second fuzzy sets B_i, the derivation follows the same scheme as above and we get

$$\frac{\partial \tilde{Q}}{\partial B_l(y')} = -2 \sum_{x\in\mathbf{X}} \left(R(x,y') - \bigvee_{i=1}^{c} \left(A_i(x) \wedge B_i(y') \right) \right) \cdot \varphi\left(A_l(x) \wedge B_l(y'), \bigvee_{i=1, i\neq l}^{c} \left(A_i(x) \wedge B_i(y') \right) \right) \cdot \psi\left(B_l(y'), A_l(x) \right) \tag{10}$$

By using this traditional method in [11], we observe that if the learning rate α is higher, the above process can be completed for a shorter computation time but the result of the approximation is not good. If the learning rate α is lower, the same process requires larger computation time and the result of approximation is better. Figure 1 shows an overview of the decomposition of the fuzzy relation.

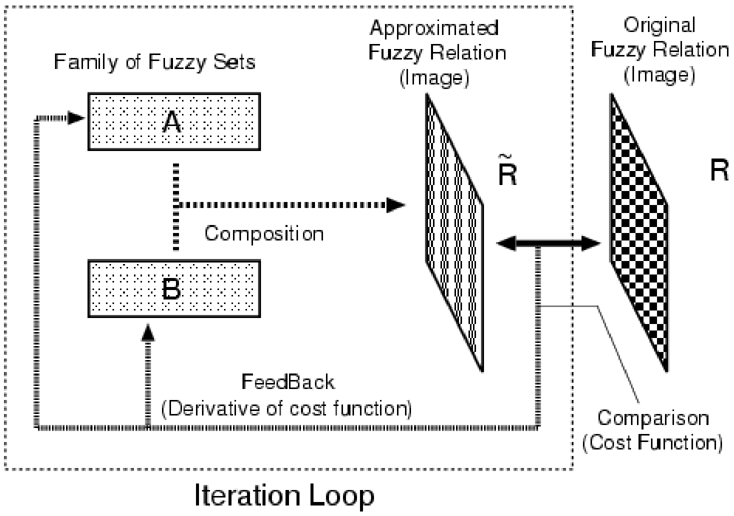

Figure 1 Overview of Decomposition of Fuzzy Relation

In order to reduce the computation time to solve the problem, [9] have shown that the derivative (6) - (10) are approximately transformed into the following equations:

$$\frac{\partial \tilde{Q}}{\partial A_l(x')} = -2\sum_{y\in \mathbf{Y}}\left(R(x',y) - \bigvee_{i=1}^{c}\left(A_i(x') \wedge B_i(y)\right)\right)\cdot \phi\left(A_l(x'), \tilde{R}(x',y)\right), \tag{11}$$

$$\frac{\partial \tilde{Q}}{\partial B_l(y')} = -2\sum_{x\in \mathbf{X}}\left(R(x,y') - \bigvee_{i=1}^{c}\left(A_i(x) \wedge B_i(y')\right)\right)\cdot \phi\left(B_i(y'), \tilde{R}(x,y')\right), \tag{12}$$

$$\phi(a,b) = \begin{cases} 1 & \text{if} \quad a = b, \\ 0 & \quad \text{otherwise.} \end{cases} \tag{13}$$

The complexities of the conventional method (Eqs. (6)-(10)) and the proposed method (Eqs. (11)-(13)) correspond to

$$\text{Conventional} = c^2 (T + C)(M + N), \tag{14}$$

$$\text{Proposed} = cC(M + N)$$

(15)

where T and C express the computation time of the minimum "$\wedge$" and comparison operation for defining the functions (7), (8), and (13), respectively. From a theoretical point of view, the complexities decrease approximately of about $1/c$.

Example)

With respect to the original images "Text" (Figure 2) and "Texture" (Figure 6), the component fuzzy sets and their approximated fuzzy relations are obtained as Figures 3, 4, 5, 7, 8, 9.

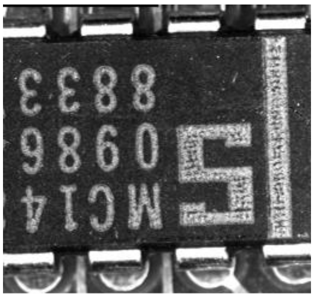

Figure 2. Original fuzzy relation "Text"

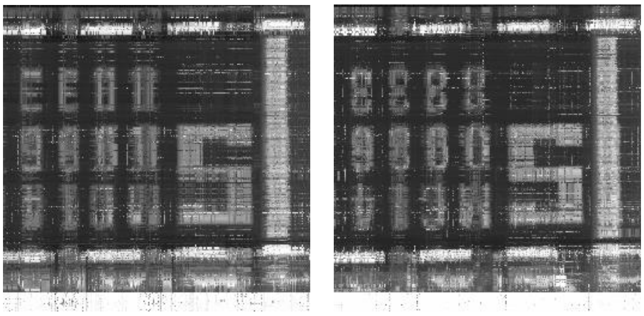

Figure 3 Approximated fuzzy relation (left : c = 50, Q = 1500.41),(right : c = 100, Q=1261.63)

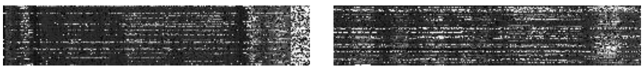

Figure 4. Decomposed fuzzy sets A (left) and B (right), Schein rank = 50

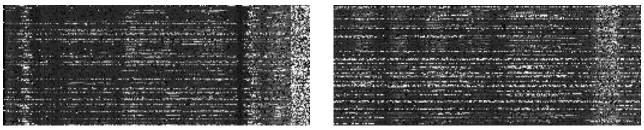

Figure 5. Decomposed fuzzy sets A (left) and B (right), Schein rank = 100

Figure 6. Original fuzzy relation “Texture”

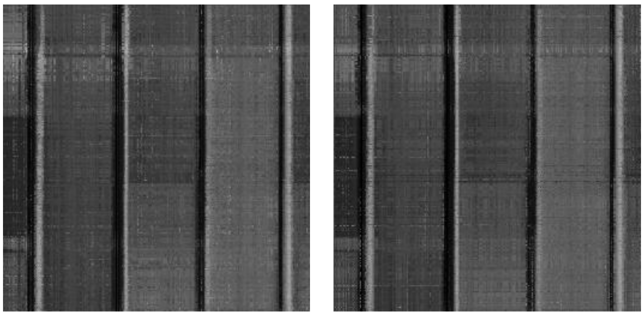

Figure 7. Approximated fuzzy relation (left : c = 50, Q = 381.48),(right : c = 100, Q=351.14)

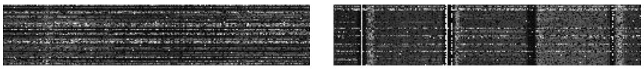

Figure 8. Decomposed fuzzy sets A (left) and B (right), Schein rank = 50

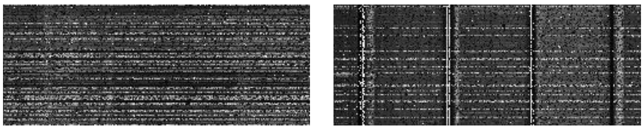

Figure 9. Decomposed fuzzy sets A (left) and B (right), Schein rank = 100

In the case of decomposition of the original image, we need large computation time to solve the decomposition problem and the decomposed results are sometimes too fine. This paper proposes the decomposition of fuzzy relation based on the multi-resolution schemes, in order to reduce the computational time and get the global feature of the original image as shown in Figure 10. The proposed method employs the ICF (image compression method based on fuzzy relational equations [3][5]) as multi-resolution scheme, and it can produce the image compression with the restricted region. Therefore, the proposed method can perform the analysis of the restricted regions of the original image as shown in Figure 11. The detailed explanation will be given in next section.

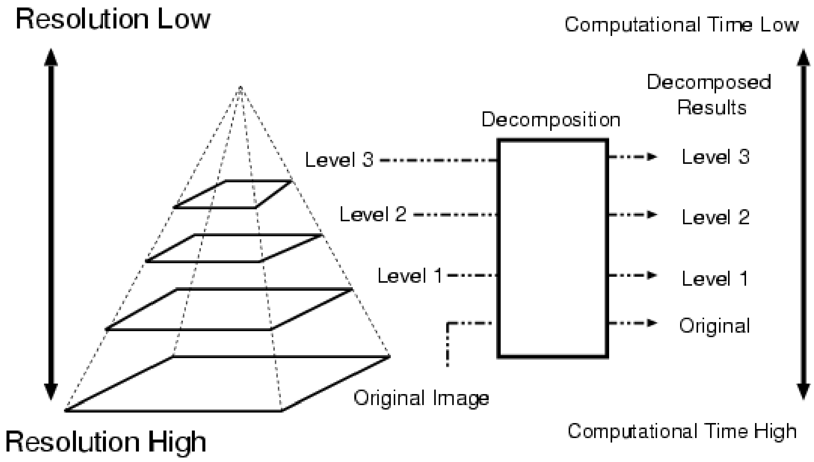

Figure 10 Correspondence of resolution of image and computational time of decomposition problem

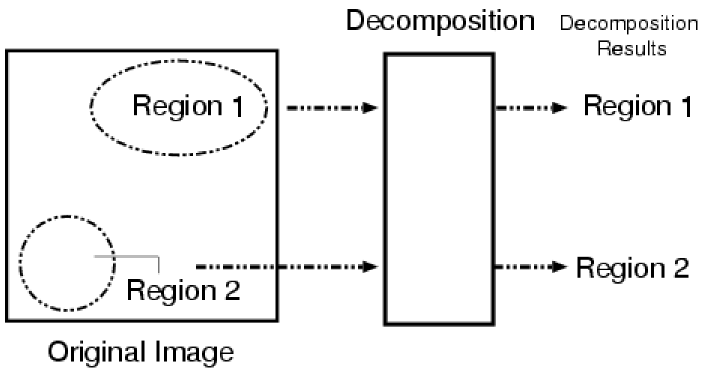

Figure 11 Decomposition of restricted regions of the original image

3. Decomposition of Fuzzy Relation Based on Multi-Resolution Scheme

This paper proposes a decomposition of fuzzy relation based on multi-resolution scheme. An overview of the proposed method is shown in Figure 12. In the proposed method, the original image R is transformed (compressed) into the compressed image G by the operation $\theta^{\downarrow}(\ )$. Afterward, the decomposition problem shown in Sec.2 is applied to the compressed image G. By solving the decomposition problem of G, component fuzzy sets $\mathbf{A}^{(\mathrm{G})}$ and $\mathbf{B}^{(\mathrm{G})}$ can be obtained. The reconstructed image $\hat{\mathrm{R}}$ is obtained by an operation $\theta^{\uparrow}(\ )$, where $\tilde{\mathrm{G}}$ is given by the composition of the component fuzzy sets $\mathbf{A}^{(\tilde{\mathrm{G}})}$ and $\mathbf{B}^{(\tilde{\mathrm{G}})}$.

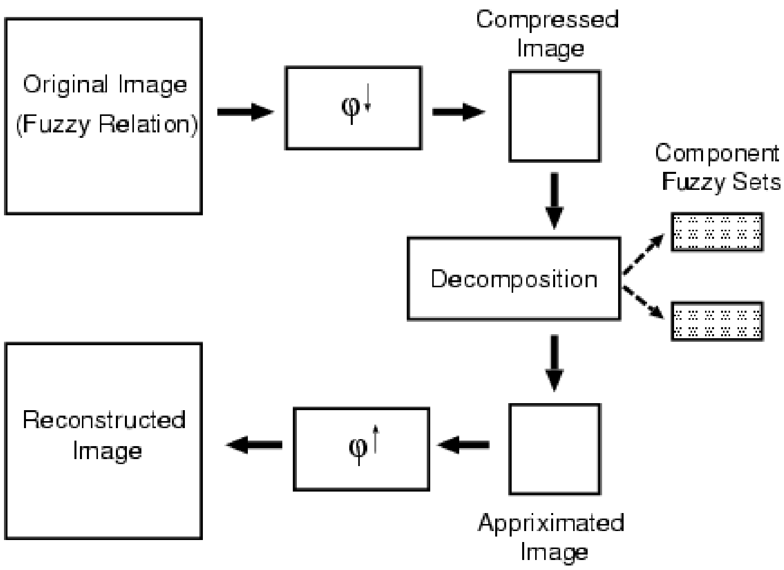

Figure 12 An overview of the proposed method

In this paper, the operation $\theta^{\downarrow}(\ )$ and $\theta^{\uparrow}(\ )$ are defined as

$$G = \theta^{\downarrow}(R), \tag{16}$$

$$G(i,j) = \max_{y \in \mathbf{Y}} \left\{ D_j(y) \, t \max_{x \in \mathbf{X}} \left\{ C_i(x) \, t R(x,y) \right\} \right\}, \tag{17}$$

$$C = \{C_1, C_2, \ldots, C_I\}, \tag{18}$$

$$C_i(x_m) = \mathbf{exp}\left(-Sh\left(\frac{iM}{I} - m \right)^2 \right), \ m = 1, 2, \ldots, M, \qquad 19)$$

$$D = \{D_1, D_2, \ldots, D_J\}, \tag{20}$$

$$D_j(y_n) = \mathbf{exp}\left(-Sh\left(\frac{jN}{J} - n \right)^2 \right), \ n = 1, 2, \ldots, N \tag{21}$$

$$\tilde{R} = \theta^{\uparrow}(\tilde{G}), \tag{22}$$

$$\tilde{R}(x,y) = \min_{j \in \mathbf{J}} \left\{ D_j(y) \alpha_t \min_{i \in \mathbf{I}} \left\{ C_i(x) \alpha_t \tilde{G}(i,j) \right\} \right\}, \tag{23}$$

where α_t denotes the t-relative pseudo-complement defined by

$$a\alpha_t b = \mathbf{sup}\left\{c \in \left[0,1\right] \mid atb \le c\right\}. \tag{24}$$

Figure 13 shows the original image "Lena" and the compressed images obtained by the ICF of the size 200 x 200, 100 x 100, 50 x 50, and 30 x 30, respectively. By applying the decomposition method to these compressed images, the global feature of the original image can be obtained by lower computational time than that of the decomposition of the original image.

Figure 13 Original image "Lena" (Left) and the compressed images 200 x 200, 100 x 100, 50 x 50, and 30 x 30, respectively.

Furthermore, the ICF can produce the image compression restricted to an image region by non-uniform coders as shown in Figure 14. By applying the decomposition method to the restricted compressed image, we can obtain the component analysis of the restricted region. These analysis results are used for the image retrieval and computer aided diagnosis.

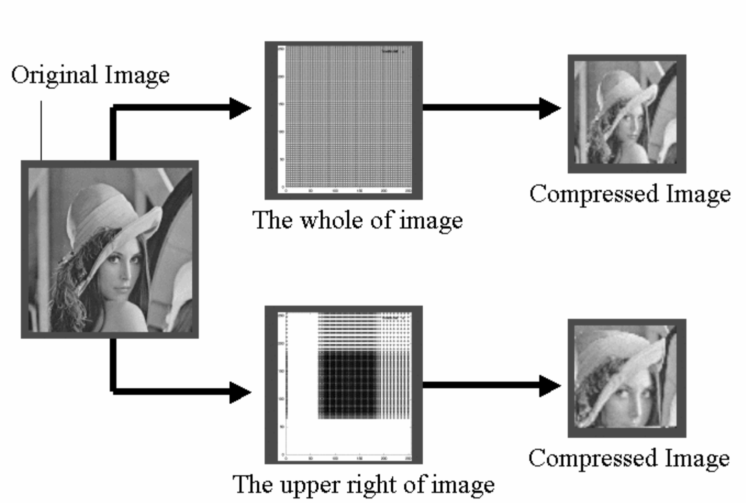

Figure 14 Image Compression restricted to an image region

4 Experiments

In order to confirm the effectiveness of the proposed method, an experimental comparison is performed. The test image “Lena” of the size 256 x 256 is shown in Figure 15. In the case of the proposed method, the size of the compressed fuzzy relation is 128 x 128. Under the condition that the Schein rank is 50 and 100, respectively, we perform the decomposition by the conventional method [9] and the proposed one. A comparison of the computation time is shown in Tab. 1, and it is confirmed the effectiveness of the proposed method and the quality of the approximated fuzzy relation obtained by the proposed method is comparable to that of the conventional one.

Figure 15 Original image (Lena)

Tab. 1 : Computation time comparison

Schein Rank	*Computation time*	
	Conventional [9]	*Proposed*
50	*132.98 (s)*	*46.82 (s)*
100	*343.64 (s)*	*138.42 (s)*

The approximated fuzzy relations and the component fuzzy sets with Schein rank = 50 and 100 obtained by the conventional method are shown in Figs. 16, 17, and 18.

Figures 19, 20, 21, and 22 shows the approximated fuzzy relations and the component fuzzy sets with Schein rank = 50 and 100 obtained by the conventional method.

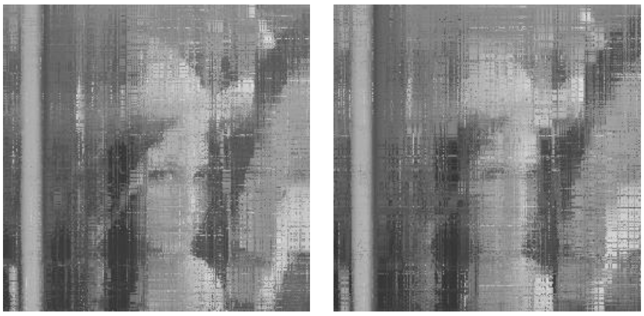

Figure 16 Approximated fuzzy relations obtained by conventional method (left: c = 50, RMSE = 16.825), (right: c=100, RMSE = 16.263)

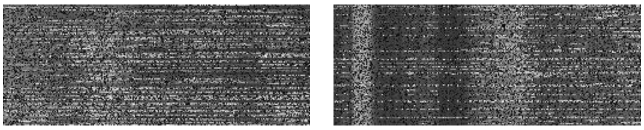

Figure 17 Decomposed fuzzy sets A (left) and B (right) obtained by conventional method, c = 100

Figure 18 Decomposed fuzzy sets A (left) and B (right) obtained by conventional method, c = 50

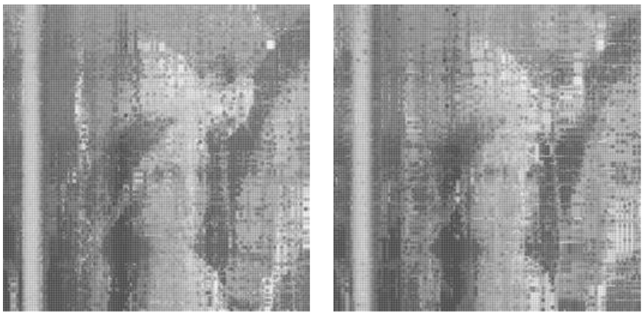

Figure 19 Approximated fuzzy relations obtained by proposed method, (left: c = 100, RMSE = 22.372), (right: c = 50, left, RMSE = 22.724)

Figure 20 Original compressed fuzzy relation (left), approximated compressed fuzzy relations obtained by proposed method, c = 50 (middle), c = 100 (right).

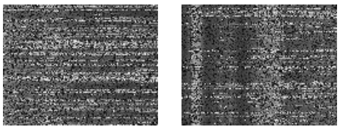

Figure 21 Decomposed fuzzy sets A (left) and B (right), c = 100

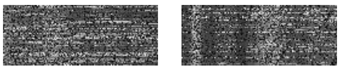

Figure 22 Decomposed fuzzy sets A (left) and B (right), c = 50

5 Conclusions

This paper has proposed a decomposition method of fuzzy relation based on multi- resolution scheme. The proposed method treats with the target image as the compressed image obtained by a compression and reconstruction method by fuzzy relational equations (ICF) [3][5], therefore the computation time can be reduced compared with the decomposition of the original one. In the proposed method, by solving the decomposition problem of the compressed image instead of the original one, the global feature of the original image as the component fuzzy sets can be obtained. Furthermore, the ICF can produce the image compression restricted to an image region, that is, the component analysis of the region selected by users can be performed by the proposed method. Particularly, the ICF can achieve the image restriction by non-uniform coders that correspond to families of fuzzy sets expressing the human subjectivity.

In an experiment using the test images extracted from Standard Image DataBAse (SIDBA), it was confirmed that the computation time of the proposed method is decreased into 31.4% of the conventional one, under the condition that the size of original image and compressed image is 256 x 256 and 128 x 128, respectively.

The component analysis results obtained by the proposed method can be used for the image retrieval and computer aided diagnosis based on medial images, and these topics will be future studies.

References

[1] A. Di Nola, W. Pedrycz, and S. Sessa: *When is a fuzzy relation decomposable in two fuzzy sets?*, Fuzzy Sets and Systems, vol. 16, pp. 87-90, 1985.

[2] A. Di Nola, W. Pedrycz, E. Sanchez, and S. Sessa: *Fuzzy Relation Equations and Its Applications to Knowledge Engineering*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1989.

[3] K. Hirota, and W. Pedrycz: *Fuzzy Relational Compression*, IEEE Transactions on Systems, Man, and Cybernetics, Part B, vol. 29, No. 3, pp. 407-415, 1999.

[4] K. H. Kim and F. W. Roush: *Generalized Fuzzy Matrices*, Fuzzy Sets and Systems, vol. 4, pp. 293-315,1980.

[5] H. Nobuhara, W. Pedrycz, and K. Hirota: *Fast Solving Method of Fuzzy Relational Equation and its Application to Image Compression/Reconstruction*, IEEE Transaction on Fuzzy Systems, vol. 8, No. 3, pp. 325-334, 2000.

[6] H. Nobuhara, Y. Takama, and K. Hirota: *Fast Iterative Solving Method of Various Types of Fuzzy Relational Equations and its Application to Image Reconstruction*, International Journal of Advanced Computational Intelligence, vol. 5, No. 2, pp. 90-98, 2001.

[7] H. Nobuhara, Y. Takama, and K. Hirota: *Fast Iterative Solving Method of Fuzzy Relational Equation and its Application to Image Compression/Reconstruction*, International Journal of Fuzzy Logic and intelligent Systems, vol. 2, No. 1, pp. 38-42, 2002.

[8] H. Nobuhara, W. Pedrycz, and K. Hirota: *A Digital Watermarking Algorithm Using Image Compression Method Based on Fuzzy Relational Equation,* Proceedings of the 2002 IEEE International Conference on Fuzzy Systems, vol. 2, pp. 1568-1573, 2002.

[9] H. Nobuhara, K. Hirota, W. Pedrycz, and S. Sessa: *Two Iterative Methods of Decomposition of a Fuzzy Relation for Image Compression/Decomposition Processing*. Soft Computing, to appear.

[10] W. Pedrycz: *Optimization Schemes for Decomposition of Fuzzy Relations,* Fuzzy Sets and Systems, vol. 100, pp. 301-325, 1998.

[11] W. Pedrycz, K. Hirota, and S. Sessa: *A Decomposition of Fuzzy Relations*, IEEE Transaction on Systems, Man, and Cybernetics, Part B, vol. 31, No. 4, pp. 657-663, 2001.