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Fuzzy Sublattices and Fuzzy Ideals

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Abstract: In this paper the idea of *fuzzy subsets* of *lattices* has been applied to develop different types of *fuzzy sublattice* structures. We have defined two *binary operations* $\circ$ and $*$ on the *set* of *fuzzy subsets* of a given *lattice*, and obtained interesting results relating to these *binary compositions*. We have introduced the concepts of *fuzzy sublattices* and *fuzzy ideals* of *lattices*, and some basic results relating to these concepts have been proved. The *images* and *pre-images* of *fuzzy sublattices* and *fuzzy ideals* under *lattice-homomorphisms* have been studied.

1. Introduction

Throughout this paper L always denotes a *lattice*. The *meet*, *join* and *partial ordering* of L, will be denoted by $\wedge, \vee$ and $\leq$, respectively. The closed interval [0,1] has been denoted by *I*.

Definition 1.1. By a *fuzzy subset* of L, we mean a *function* from L into [0.1].

The *set* of all *fuzzy subsets* of *L* is denoted by $I^L = [0,1]^L$.

Definition 1.2. Let $X \subseteq L$ and $t \in (0,1]$, we define $X_t \in I^L$ as follows **:**

$$X_t(x) = t, \text{ for } x \in X$$

$$X_t(x) = 0, \text{ for } x \in L \sim X .$$

In particular, if X is a *singleton*, say, $\{x\}$, then $\{x\}_t$ is often referred to as a *fuzzy singleton*, and denoted by x_t.

Definition 1.3.[2] A *fuzzy subset* μ of a *lattice* L is said to have *inf property* if for any subset *S* of L, there exists some $s_1 \in S$ such that $\mu(s_1) = \sup\{\mu(s) : s \in S\}$.

Definition 1.4. We define two *binary operations* $\circ$ and $*$ on I^L as follows **:**
$\forall\, \mu, \sigma \in I^L, \forall\, x \in L$,

$$(\mu \circ \sigma)(x) = \sup\{\mu(y), \sigma(z)\} : y, z \in L, \mathrm{x} = y \wedge z \text{ and}$$
$$(\mu * \sigma)(x) = \sup\{\mu(y), \sigma(z)\} : y, z \in L, \mathrm{x} = y \vee z$$

We state the following results whose proofs are simple.

Theorem 1.5 Let $\mu, \mu', \sigma, \sigma' \in I^L$. Let $\mathrm{t} = \sup\{\mu(x) : x \in L\}$. Then $\forall\, x \in L$,

(1) $(\mu \circ \sigma)(x) \geq \sup\{\min(\mu(x), \sigma(x \vee y), \min(\mu(x \vee y), \sigma(x)) : y \in L\}$;

(2) $(\mu * \sigma)(x) \geq \sup\{\min(\mu(x), \sigma(x \wedge y), \min(\mu(x \wedge y), \sigma(x)) : y \in L\}$;

(3) $\mu \circ \sigma = \sigma \circ \mu$ and $\sigma * \mu = \mu * \sigma$;

(4) $\mu \subseteq \mu'$ and $\sigma \subseteq \sigma'$ implay $\mu \circ \sigma \subseteq \mu' \circ \sigma'$ and $\mu * \sigma \subseteq \mu' * \sigma'$;

(5) $\mu \cap \sigma \subseteq \mu \circ \sigma$ and $\mu \cap \sigma \subseteq \mu * \sigma$;

(6) $\mu \subseteq \mu \circ \mu$ and $\mu \subseteq \mu * \mu$;

(7) $(\mu \circ x_t)(y) \geq \mu(y) \ \forall \ y \in L$ if $x \geq y$;

(8) $(\mu * x_t)(y) \geq \mu(y) \ \forall \ y \in L$ if $y \geq x$.

Following example shows that $\mu \circ \sigma \neq \mu * \sigma$.

Example 1.6. Consider the following *lattice* $L = \{a, b, c, d\}$, where $a > c > d$ and $a > b > d$; and b, c are *noncomparable*.

Define $\mu, \sigma \in I^L$ as follows**:** $\mu(a) = .5$, $\mu(b) = .7$, $\mu(c) = .3$, $\mu(d) = .3$, and $\sigma(a) = .4$, $\sigma(b) = .4$, $\sigma(c) = .6$, and $\sigma(d) = .6$. It can be verified that $\mu \circ \sigma \neq \mu * \sigma$.

2. Fuzzy Sublattices

In this section we define *fuzzy sublattice* of a *lattice* and propose to prove some fundamental theorems.

Definition 2.1. A *fuzzy subset* μ of a *lattice* L is said to be a *fuzzy sublattice* of L if $\forall \ a, b \in L$,

$\mu(a \wedge b) \geq \min(\mu(a), \mu(b))$, (i) and

$\mu(a \vee b) \geq \min(\mu(a), \mu(b))$ (ii)

The *set* of all *fuzzy sublattices* of L is denoted by $I(L)$.

Lemma 2.2 Let $\mu \in I^L$. Then μ is a *fuzzy sublattice* of L if and only if μ_t is a *sublattice* of $L \ \forall \ t \in \mathrm{Im}(\mu)$.

Proof. Suppose μ is a *fuzzy sublattice* of L. Let $t \in \mathrm{Im}(\mu)$. Then there exists $a \in L$ such that $\mu(a) = t$ and so $a \in \mu_t$. Thus $\mu_t \neq \phi$. Let $x, y \in \mu_t$. Then $\mu(x) \geq t, \mu(y) \geq t$. Since μ is a *fuzzy sublattice* of L, $\mu(x \wedge y) \geq \min(\mu(x), \mu(y)) \geq t$, and $\mu(x \vee y) \geq \min(\mu(x), \mu(y)) \geq t$. Thus $x \wedge y \in \mu_t$ and $x \vee y \in \mu_t$. Hence μ_t is a *sublattice* of L. Conversely, suppose that μ_t is a *sublattice* of L for every $t \in \mathrm{Im}(\mu)$. Let $x, y \in L$ and $t = \min(\mu(x), \mu(y))$. Then $t \in \mathrm{Im}(\mu)$ and $x, y \in \mu_t$.

Since μ_t is a *sublattice* of L, $x \wedge y \in \mu_t$ and $x \vee y \in \mu_t$.

Hence $\mu(x \wedge y) \geq t = \min(\mu(x), \mu(y))$ and $\mu(x \vee y) \geq t = \min(\mu(x), \mu(y))$.

Thus μ is a *fuzzy sublattice* of L.

Example 2.3. Consider the following *lattice* $L = \{a, b, c, d, e\}$, where $a > c > e$ and $a > b > d > e$; and *b*, *c* and *c*, *d* are *noncomparable*.

A *fuzzy subset* μ of L is defined as follows **:**

$\mu(a) = .6$, $\mu(b) = .1$, $\mu(c) = .5$, $\mu(d) = .6$, $\mu(e) = .7$.

Here $\mathrm{Im}(\mu) = \{.1, .5, .6, .7\}$. $\mu_{.1} = L$, $\mu_{.5} = \{a, c, d, e\}$, $\mu_{.6} = \{a, d, e\}$, $\mu_{.7} = \{e\}$.

We observe that μ_t is a *sublattice* of $L \; \forall \, t \in \mathrm{Im}(\mu)$.

Hence μ is a *fuzzy sublattice* of L.

We state the following theorem whose is simple.

Theorem 2.4 If μ is a *fuzzy sublattice* of L, then $\mu^* = \{x \in L : \mu(x) > 0\}$ is a *sublattice* of L.

Theorem 2.5 Any fuzzy subset of a chain L is a fuzzy sublattice of L.

Proof. Let L be a *chain* and μ be a *fuzzy subset* of L. Let $x, y \in L$.

Then either $x \geq y$ or $y \geq x$. If $x \geq y$, then $x \vee y = x$ and $x \wedge y = y$. Therefore we have $\mu(x \wedge y) = \mu(y) \geq \min(\mu(x), \mu(y))$ and $\mu(x \vee y) = \mu(x) \geq \min(\mu(x), \mu(y))$.

Hence μ is a *fuzzy sublattice* of L.

On the other hand, if $y \geq x$, then $x \vee y = y$ and $x \wedge y = x$. Therefore we have $\mu(x \wedge y) = \mu(x) \geq \min(\mu(x), \mu(y))$, and $\mu(x \vee y) = \mu(y) \geq \min(\mu(x), \mu(y))$. Hence μ is a *fuzzy sublattice* of L.

Theorem 2.6 Intersection of any number of fuzzy sublattices of L is a fuzzy sublattice of L.

Proof. Let $\{\mu_i\}$ be any collection of *fuzzy sublattices* of *L*. Suppose $x, y \in L$.

Then $(\cap \mu_i)(x \wedge y) = \inf\{\mu_i(x \wedge y)\} \geq \inf\{\min(\mu_i(x), \mu_i(y))\} =$

$= \min(\inf\{\mu_i(x)\}, \inf\{\mu_i(y)\}) \qquad = \min((\cap \mu_i)(x), (\cap \mu_i)(y))$,

and $(\cap \mu_i)(x \vee y) = \inf\{\mu_i(x \vee y)\} \geq \inf\{\min(\mu_i(x), \mu_i(y))\} =$

$= \min(\inf\{\mu_i(x)\}, \inf\{\mu_i(y)\}) \; = \min((\cap \mu_i)(x), (\cap \mu_i)(y))$. Hence $\cap \mu_i$ is a *fuzzy sublattice* of *L*.

Following example shows that the *union* of two *fuzzy sublattices* of *L* need not be a *fuzzy sublattice* of *L*.

Example 2.7 Consider the following *lattice* $L = \{a, b, c, d\}$, where $a > c > d$ and $a > b > d$; and *b, c* are *noncomparable*.

Define $\mu, \sigma \in I^L$ as follows **:** $\mu(a) = .5$, $\mu(b) = .7$, $\mu(c) = .3$, $\mu(d) = .3$, and $\sigma(a) = .4$, $\sigma(b) = .4$, $\sigma(c) = .6$, and $\sigma(d) = .6$. It can be verified that $\mu \cup \sigma$ is not a fuzzy sublattice of *L*.

Following example shows that the *union* of two *fuzzy sublattices* of *L* may be a *fuzzy sublattice* of *L*.

Example 2.8 Consider the *lattice* *L* in Example 2.7. Define $\mu, \sigma \in I^L$ as follows**:** $\mu(a) = .5$, $\mu(b) = .5$, $\mu(c) = .3$, $\mu(d) = .3$, and $\sigma(a) = .4$, $\sigma(b) = .4$, $\sigma(c) = .6$, and $\sigma(d) = .6$.

Here $\text{Im}(\mu) = \{.3,.5\}$ and $\mu_{.3} = L$, $\mu_{.5} = \{a, b\}$ are all *sublattices* of *L*. Hence μ is a *fuzzy sublattice* of *L*. $\text{Im}(\sigma) = \{.4,.6\}$, and $\sigma_{.4} = L$, $\sigma_{.6} = \{c, d\}$ are *sublattices* of *L*. Hence σ is a *fuzzy sublattice* of *L*.

Thus $\text{Im}(\mu \cup \sigma) = \{.5,.6\}$. We find that $(\mu \cup \sigma)_{.6} = \{c, d\}$, and $(\mu \cup \sigma)_{.5} = L$ are all *sublattices* of *L*. Hence $\mu \cup \sigma$ is a *fuzzy sublattice* of *L*.

Theorem 2.9 A *fuzzy subset* μ of *L* is a *fuzzy sublattice* of *L* if and only if $\mu \circ \mu \subseteq \mu$ and $\mu * \mu \subseteq \mu$.

Proof. Suppose μ is a *fuzzy subset* of *L*. We assume that $\mu \circ \mu \subseteq \mu$ and $\mu * \mu \subseteq \mu$. Then $\forall\, x, y \in L, \mu(x \wedge y) \geq (\mu \circ \mu)(x \wedge y) \geq \min(\mu(x), \mu(x))$, since $(\mu \circ \mu)(a) = \sup\{\min(\mu(x), \mu(y)) : x, y \in L, a = x \wedge y\}$ implies $(\mu \circ \mu)(a) \geq \min(\mu(x), \mu(y))\ \forall\, x, y \in L$ such that $a = x \wedge y$.

Again, $\forall\, x, y \in L, \mu(x \vee y) \geq (\mu * \mu)(x \vee y) \geq \min(\mu(x), \mu(y))$, since $(\mu * \mu)(a) = \sup\{\min(\mu(x), \mu(y)) : x, y \in L, a = x \vee y\}$ implies $(\mu \circ \mu)(a) \geq \min(\mu(x), \mu(y))\ \forall\, x, y \in L$ such that $a = x \vee y$. Thus we find that $\forall\, x, y \in L$, $\mu(x \wedge y) \geq \min(\mu(x), \mu(y))$, and $\mu(x \vee y) \geq \min(\mu(x), \mu(y))$. Hence μ is a *fuzzy sublattice* of *L*.

Conversely, suppose that μ is a *fuzzy sublatice* of *L*. Let $x \in L$ and $x = a \wedge b$, where $a, b \in L$. Then $\mu(x) = \mu(a \wedge b) > \min(\mu(a),\ \mu(b))$. Thus $\mu(x) \geq \min(\mu(a),\ \mu(b))\ \ \forall a, b \in L$ such that $x = a \wedge b$.

Hence $\mu(x) \geq \sup\{\min(\mu(a), \mu(b)) : a, b \in L, x = a \wedge b\} = (\mu \circ \mu)(x)$. Therefore $\mu \circ \mu \subseteq \mu$. On the other hand if $x \in L$ and $x = a \vee b$; $a, b \in L$, then $\mu(x) = \mu(a \vee b) \geq \min(\mu(a), \mu(b))$. Thus $\mu(x) \geq \min(\mu(a), \mu(b)), \forall\, a, b \in L$ such that $x = a \vee b$. Hence $\mu(x) \geq \sup\{\min(\mu(a), \mu(b)) : a, b \in L, x = a \vee b\} = (\mu * \mu)(x)$. Thus we find that $\mu * \mu \subseteq \mu$. This completes the proof.

Corollary 2.10 A *fuzzy subset* μ of *L* is a *fuzzy sublattice* of *L* if and only if $\mu \circ \mu = \mu$ and $\mu * \mu = \mu$.

Proof. The proof is trivial

3. Images and Inverse images of Fuzzy Sublattices Under Lattice – Homomorphisms

Definition 3.1.[1] A *mapping f* from a *lattice L* into a *lattice M* is said to be a *lattice-homomorphism* if $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$, $\forall\, x, y \in L$.

Theorem 3.2. Let *f* be a *lattice –epimorphism* of a *lattice L* onto a *lattice M*. If $\mu \in I(L)$, then $f(\mu) \in I(M)$.

Proof. Let $\mu \in I(L)$. Suppose $u, v \in M$. Then $u = f(x)$, $v = f(y)$ for some $x, y \in L$, and so $u \wedge v = f(x) \wedge f(y) = f(x \wedge y)$ and $u \vee v = f(x) \vee f(y) = f(x \vee y)$. Now, $f(u)(u \wedge v) = \sup\{\mu(z) : z \in L,\ f(z) = u \wedge v\} \geq \sup\{\mu(x \wedge y) : x, y \in L,\ f(x) = u,\ f(y) = v\} \geq \sup\{\min(\mu(x), \mu(y)) : x, y \in L,\ f(x) = u,\ f(y) = v\} = \min(\sup\{\mu(x) : x \in L,\ f(x) = u\}), \sup\{\mu(y) : y \in L,\ f(y) = v\}) = \min\{f(\mu)(u), f(\mu)(v)\}$, and $f(\mu)(u \vee v) = \sup\{\mu(z) : z \in L,\ f(z) = u \vee v\} \geq \sup\{\mu(x \vee y) : x, y \in L,\ f(x) = u,\ f(y) = v\} \geq f(x) = u,\ f(y) = v\} = \min(\sup\{\mu(x) : x \in L, f(x) = u\}, \sup\{\mu(y) : y \in L,\ f(y) = v\}) = \min\{f(\mu)(u), f(\mu)(v)\}$.

Hence $f(\mu)$ is a *fuzzy sublattice* of *M*, i.e., $f(\mu) \in I(M)$.

Theorem 3.3 Let *f* be a *lattice –homomorphism* of a *lattice L* into a *lattice M*. If $v \in I(M)$ then $f^{-1}(v) \in I(L)$.

Proof. Let $v \in I(M)$. Suppose $x, y \in L$. Then $f^{-1}(v)(x \wedge y) = v(f(x \wedge y)) = v(f(x) \wedge f(y)) \geq \min(v(f(x) \vee f(y))) \geq \min(v(f(x)), v(f(y))) = = \min\{f^{-1}(v)(x),\ f^{-1}(v)(y)\}$ and $f^{-1}(v)(x \vee y) = v(f(x \vee y)) = v(f(x) \vee f(y)) \geq \min(v(f(x)), v(f(y))) = \min\{f^{-1}(v)(x), f^{-1}(v)(y)\}$.

Hence $f^{-1}(v)$ is a *fuzzy sublattice* of *L*, i.e., $f^{-1}(v) \in I(L)$.

4. Fuzzy Sublattices of Complete Lattices

A lattice L is said to be complete if each of its subsets has an infimum and a supremum in L. Any nonempty complete lattice contains a least element $0 = \inf L$ and a greatest element $1 = \sup L$. A subset S of a complete lattice L is said to be a sublattice of the complete lattice if for any subset H of S, inf H and supH also belong to S.

Definition 4.1. A *fuzzy subset* μ of a *complete lattice* L is said to be a *fuzzy* C *sublattice* of L if for any *subset S* of L,

$\mu(\vee\{x_\alpha \in S\}) \geq \inf\{\mu(x_\alpha) : x_\alpha \in S\}$ (iii) and

$\mu(\wedge\{x_\alpha \in S\}) \geq \inf\{\mu(x_\alpha) : x_\alpha \in S\}$ (iv)

Theorem 4.2. Suppose μ is a *fuzzy subset* of a *complete lattice* L and has the *inf. property*. Then μ is a *fuzzy C sublattice* of L if and only if μ_t is a *sublattice* of L, $\forall\, t \in \mathrm{Im}(\mu)$.

Proof. Let μ be a *fuzzy C sublattice* of the *complete lattice* L. Let $t \in \mathrm{Im}(\mu)$ and $S \subseteq \mu_t$. Then $\mu(x_\alpha) \geq t$, $\forall\, x_\alpha \in S$. Now, $\mu(\vee\{x_\alpha \in S\}) \geq$ $\inf\{\mu(x_\alpha) : x_\alpha \in S\} \geq t\}$, and $\mu(\wedge\{x_\alpha \in S\}) \geq \inf\{\mu(x_\alpha) : x_\alpha \in S\} \geq t\}$.

Thus $\vee\{x_\alpha : x_\alpha \in S\} = \sup S \in \mu_t$ and $\wedge\{x_\alpha : x_\alpha \in S\} = \inf S \in \mu_t$.

Hence μ_t is a *sublattice* of $L\ \forall\, t \in \mathrm{Im}(\mu)$.

Conversely, suppose that μ_t is a *sublattice* of the complete lattice L $\forall\, t \in \mathrm{Im}(\mu)$. Let $S \subseteq L$ and $\inf\{\mu(x_\alpha) : x_\alpha \in S\} = t$.

Since μ has the *inf. property,* there is *x* in *S* such that $\mu(x) = t$, and so $x \in \mu_t$. Thus $t \in \mathrm{Im}(\mu)$ and μ_t is a nonempty *subset* of L. Since $\mu(x_\alpha) \geq t$, $\forall\, x_\alpha \in S$, it follows that $S \subseteq \mu_t$, and so $\wedge\{x_\alpha : x_\alpha \in S\} \in \mu_t$ and $\vee\{x_\alpha : x_\alpha \in S\} \in \mu_t$. Hence $\mu(\vee\{x_\alpha \in S\}) \geq t = \inf\{\mu(x_\alpha) : x_\alpha \in S\}$, and $\mu(\wedge\{x_\alpha \in S\}) \geq t = \ = \inf\{\mu(x_\alpha) : x_\alpha \in S\}$.

Thus μ is a *fuzzy C sublattice* of L. This completes the proof.

5. Fuzzy Ideals

Definition 5.1.[1] Let L be a *lattice*. A nonempty *subset J* of L is called an *ideal* of L if the following properties hold :

$a \in J, x \in L, x \leq a \Rightarrow x \in J$ (v)

$a \in J, b \in J \Rightarrow a \vee b \in J$ (vi)

Definition 5.2. A *fuzzy subset* μ of a *lattice* L is called a *fuzzy ideal* of L if $\forall\, x, y \in L$, $\mu(x \wedge y) \geq \max\{\mu(x), \mu(y)\}$ and $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$.

From the definitions of *fuzzy ideal* and *fuzzy sublattice* of a *lattice* it clearly follows that every *fuzzy ideal* is a *fuzzy sublattice*. The converse is however not true as can be seen from the following example.

Example 5.3. Consider the following *lattice* $L = \{a, b, c, d, e\}$, where

$e < c < a$ and $e < d < b < a$; and *b, c* are *noncomparable*, also *c, d* are *noncomparable*.

Define $\mu \in I^L$ as follows :

$\mu(a) = .6$, $\mu(b) = .1$, $\mu(c) = .5$, $\mu(d) = .6$, $\mu(e) = .7$. Hence $\text{Im}\mu = \{.1,.5,.6,.7\}$.

Now, $\mu_{.1} = L$, $\mu_{.5} = \{a, c, d, e\}$, $\mu_{.6} = \{a, d, e\}$, $\mu_{.7} = \{e\}$ are all *sublattices* of L. Hence μ is a *fuzzy sublattice* of L.

On the other hand we find that $\mu(a \wedge b) = \mu(b) = .1$ but $\max\{\mu(a), \mu(b\} = .6$.

Thus $\mu(a \wedge b) < \max\{\mu(a), \mu(b\}$. Hence μ is not a *fuzzy ideal* of L.

Example of a *fuzzy ideal.*

Example 5.4. Consider the following *lattice* $L = \{a, b, c, d\}$, where $a > c > d$ and $a > b > d$ and *b, c* are *noncomparable.*

Define $\mu \in I^L$ as follows : $\mu(a) = .2$, $\mu(b) = .2$, $\mu(c) = .4$, $\mu(d) = .5$.

Then μ is a *fuzzy ideal* of L.

Theorem 5.5. A *fuzzy subset* μ of a *lattice* L is a *fuzzy ideal* of L if and only if μ_t is an *ideal* of L $\forall\, t \in \text{Im}(\mu)$.

Proof. Suppose μ_t is an *ideal* of L $\forall\, t \in \text{Im}(\mu)$.

Let $x, y \in L$ and $t = \max\{\mu(x), \mu(y)\}$, $r = \min\{\mu(x), \mu(y)\}$. Then $t, r \in \text{Im}(\mu)$.

Hence by our hypothesis μ_t and μ_r are *ideals* of L.

Since $t = \max\{\mu(x), \mu(y)\}$, either $\mu(x) = t$ or $\mu(y) = t$.

If $\mu(x) = t$, then $x \in \mu_t$.

Since μ_t is an ideal of L, $x \wedge y \le x$ and $x \in \mu_t$, we have $x \wedge y \in \mu_t$.

Thus we find $\mu(x \wedge y) \ge t = \max\{\mu(x), \mu(y)\}$.

If $\mu(y) = t$, then $y \in \mu_t$.

Since μ_t is an ideal of L, $x \wedge y \le y$ and $y \in \mu_t$, we have $x \wedge y \in \mu_t$.

Thus we find $\mu(x \wedge y) \ge t = \max\{\mu(x), \mu(y)\}$.

Again, since $r = \min\{\mu(x), \mu(y)\}$, either $\mu(x) = r$ or $\mu(y) = r$.

If $\mu(x) = r$, then $\mu(y) \ge \mu(x) = r$. Thus $x, y \in \mu_r$. Since μ_r is an *ideal* of L, we have $x \vee y \in \mu_r$ and so $\mu(x \vee y) \ge r = \min\{\mu(x), \mu(y)\}$.

If $\mu(y) = r$, then $\mu(x) \ge \mu(x) = r$. Thus $x, y \in \mu_r$. Since μ_r is an *ideal* of L, we have $x \vee y \in \mu_r$ and so $\mu(x \vee y) \ge r = \min\{\mu(x), \mu(y)\}$.

Hence $\mu(x \vee y) \ge \min\{\mu(x), \mu(y)\}$ and $\mu(x \wedge y) \ge \max\{\mu(x), \mu(y)\}$ $\forall x, y \in L$. Consequently μ is a *fuzzy ideal* of L.

Conversely we assume that μ is a *fuzzy ideal* of L and $t \in \text{Im}(\mu)$.

We need to show that μ_t is an *ideal* of L. Let $x \in \mu_t$, $y \in L$ and $y \le x$.

Now, $y \le x$ implies $x \wedge y = y$ and $x \in \mu_t$ implies $\mu(x) \ge t$. Since μ is a *fuzzy ideal* of L, we have $\mu(x) = \mu(x \wedge y) \ge \max\{\mu(x), \mu(y)\} \ge \mu(x) \ge t$. Thus $y \in \mu_t$.

Hence $x \in \mu_t$, $y \in L$ and $y \le x$ implies $y \in \mu_t$.

Next let $x, y \in \mu_t$. Then $\mu(x) \ge t$, $\mu(y) \ge t$. Since μ is a *fuzzy ideal* of L, we have $\mu(x \vee y) \ge \min\{\mu(x), \mu(y)\} \ge t$. Thus $x \vee y \in \mu_t$. Hence $x, y \in \mu_t$ $\Rightarrow x \vee y \in \mu_t$.

Therefore μ_t is an *ideal* of L $\forall\, t \in \operatorname{Im}(\mu)$. This completes the proof.

6. Images and Inverse images of Fuzzy Ideals under Lattice – Homomorphisms

Theorem 6.1. Let *f* be a *lattice –epimorphism* of a *lattice* L onto a *lattice* M. If μ is a *fuzzy ideal* of L, then $f(\mu)$ is a *fuzzy ideal* of M.

Proof. Suppose μ is a *fuzzy ideal* of L. Let $u, v \in M$. Then $f(x) = u$, $f(x) = u$ for some $x, y \in L$, and so $u \wedge v = f(x) \wedge f(y) = f(x \wedge y)$ and $u \vee v = f(x) \vee f(y) = f(x \vee y)$. Now $f(\mu)(u \wedge v)$ $= \sup\{\mu(z) : z \in L,$ $f(z) = u \wedge v\} \ge \sup(\mu(x \wedge y) : x, y \in L,\ f(x) = u,\ f(y) = v) \ge$ $\ge \sup\{\max(\mu(x), \mu(y)) : x, y \in L,\ f(x) = u,\ f(y) = v\} = \max(\sup\{\mu(x)$ $: x \in L,\ f(y) = v\}) = \max(f(\mu)(u), f(\mu)(v))$. Thus $f(\mu)(u \wedge v)$ $\ge \max(f(\mu)(u), f(\mu)(v))$ $\forall\, u, v \in M$.

On the other hand,

$f(\mu)(u \vee v) = \sup\{\mu(z) : z \in L,\ f(z) = u \vee v\} \ge \sup\{\mu(x \vee y) : x, y \in L,$ $x, y \in L,\ f(x) = u,\ f(y) = v\} \ge \sup\{\min(\mu(x), \mu(y)) : x, y \in L,\ f(x) = u,$ $f(y) = v\} = \min(\sup\{\mu(x) : x \in L,\ f(x) = u\},\ \sup\{\mu(y) : y \in L,\ f(y) = v\})$ $= \min(f(\mu)(u), f(\mu)(v))$. Thus $f(\mu)(u \vee v)$ $\ge \min(f(\mu)(u), f(\mu)(v))$ $\forall\, u, v \in M$ .Hence $f(\mu)$ is a fuzzy ideal of M.

Theorem 6.2. Let *f* be a *lattice- homomorphism* of a *lattice* L into a *lattice M*. If μ is a *fuzzy ideal* of *M* then $f^{-1}(\mu)$ is a *fuzzy ideal* of L.

Proof. Let $x, y \in L$.

Then $f^{-1}(\mu)(x \wedge y)$ $\mu(f(x \wedge y))$ $= \mu(f(x) \wedge f(y))$ $\ge \max\{\mu(f(x)), \mu(f(y))\}$ $= \max\{f^{-1}(\mu)(x), f^{-1}(\mu)(y)\}$.

Also, $f^{-1}(\mu)(x \vee y)$ $=$ $\mu(f(x \vee y)) = \mu(f(x) \vee f(y))$ $\ge \min\{\mu(f(x)), \mu(f(y))\}$ $= \min\{f^{-1}(\mu)(x), f^{-1}(\mu)(y)\}$. Hence $f^{-1}(\mu)$ is a *fuzzy ideal* of L.

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