F.S.A.I., Vol. 10, Nos. 1–2, pp. 11– 14, 2004

A Fuzzy Approach to 2×2 Games Using the Expected Value of a Fuzzy Variable

R. Mathew, M.R. Kaimal

Department of Computer Science, University of Kerala, India

Abstract: The expected value of a fuzzy variable is an idea which simplifies general decision problems involving fuzzy understandings. In this paper, this idea is used for obtaining the optimal strategies for the fuzzy approach [5] to 2×2 strategic games. The present method is computationally simple, as it requires only the computation of the expected value of a fuzzy variable to achieve the optimal strategies in the game.

1 Introduction

Contributions to the field of fuzzy game theory have been pioneered by the works of Butnariu [1]. Recently, a fuzzy approach to strategic games has been proposed by Song and Kandel [5] which is more appropriate for the analysis of 2×2 strategic games. In [2], this approach is applied to the Game of Chicken. But the examples and conclusions made in [2] and [5] reveal that the method is computationally difficult to achieve the optimal strategies even for simple instances of the game. In this paper, a new and simpler method is suggested which requires only the computation of the expected value of a fuzzy variable [3] in order to achieve the optimal strategies in the game.

The expected value of a fuzzy variable is an idea [3] which permits one to think of the expected return of a decision that he takes in a general decision making situation involving fuzzy understandings. A property of the expected value of a fuzzy variable which is useful to the present discussion is given in Theorem 1.

Theorem 1: Let ξ be a fuzzy variable with expected value $E(\xi)$, then the expected value of $a\xi+b$ is $aE(\xi)+b$, for real numbers a and b.

The proof follows from the definitions [4]. The problem of the fuzzy approach to 2×2 games using the expected value of a fuzzy variable is discussed in the next section.

2 The Fuzzy Expected Value Analysis of the model

Assume the general case of payoff table as in Table 1. $A1$ and $A2$ denote the two possible actions of player 1 and $B1$ and $B2$ to that of player 2. The concept of the fuzzy model of the 2×2 game as well as the notations used in this paper, are as defined and discussed in [2]. The solution methodology is developed for player 1, but a similar analysis can be applied to the case of player 2 also.

Table 1

The General Case of Payoff Table for the 2×2 Game

		Player 2.	
		*B*1	*B*2
Player 1.	*A*1	[*a*, *e*]	[*b*, *f*]
	*A*2	[*c*, *g*]	[*d*, *h*]

Let $\Omega_i = \{(p_i, 1-p_i), 0 \le p_i \le 1\}$ be the strategy set for player *i*, for *i* = 1, 2. As in [5] player 1 assumes a fuzzy set F_1 with a membership function, say $\mu_1(\,)$ over Ω_2, in order to represent player 1's beliefs about player 2's strategic choice. The aim of player 1 is to find a strategy in Ω_1 which optimizes two of his possibly conflicting goals. The first goal is to maximize his own expected payoff and the second goal depends on his mentality. If he is cooperative and devoted he may want to maximize his opponent's payoff and if he is non-cooperative and selfish he may want to minimize his opponent's payoff. The goal functions are represented by G_{1k}, for k=1, 2. The goal functions are assumed to be real valued functions of p_1 and p_2.

Since p_2 uniquely represents the strategy $(p_2, 1-p_2)$ in Ω_2, p_2 can be thought of as a fuzzy variable with possibility distribution function $\mu_1(\,)$ (of F_1) for $0 \le p_2 \le 1$ and zero elsewhere, and conversely, any such a possibility distribution function gives rise to a fuzzy set in Ω_2. Thus in the discussion that follows, p_2 is considered as a fuzzy variable which would then mean the particular fuzzy variable arisen by the fuzzy set F_1 in Ω_2. It is also assumed that the fuzzy variables considered are normal [3]. For a particular fuzzy variable p_2, the expected value may be denoted as p_2' and it can be observed that for any such p_2, $p_2' \in [0, 1]$, since the possibility distribution function assumes zero value outside the interval [0, 1]. Note that the goal functions G_{11} and G_{12} are real valued functions of p_1 and p_2. Therefore for a fixed p_1 in [0, 1] each goal function is also a fuzzy variable, since p_2 is a fuzzy variable. The corresponding expected values may be denoted by $E_{11}(p_1)$ and $E_{12}(p_1)$ respectively. Therefore, following the multicriteria model in [5], a membership function that measures the degree of attainment of the first goal is taken as:

$$\mu_{11}(p_1) = \frac{E_{11}(p_1) - \min 1}{\max 1 - \min 1}, \tag{1}$$

where max *i* and min *i* are used to represent the maximum and minimum value achievable by the *i*-th goal function, for i = 1, 2. In the general discussion that follows it is assumed that max i -min $i \neq 0$, for i = 1, 2. Similarly, regarding his second goal the membership function may be taken as:

$$\mu_{12}(p_1) = \frac{E_{12}(p_1) - \min 2}{\max 2 - \min 2} \quad or \quad \frac{E_{12}(p_1) - \max 2}{\min 2 - \max 2} \qquad (2)$$

accordingly, the player being cooperative or non-cooperative. Now the optimal strategy for player 1 is taken as the particular p_1 in [0, 1] which maximizes the aggregated sum in (3).

$$w_1\mu_{11}(p_1) + w_2\mu_{12}(p_1) \qquad (3)$$

Now, from Table 1 one gets:

$$G_{11}(p_1, p_2) = p_1 a p_2 + p_1 b(1-p_2) + (1-p_1)cp_2 + (1-p_1)d(1-p_2)$$
$$= p_2[p_1(a-b-c+d)+c-d] + p_1(b-d)+d \qquad (4)$$

Therefore for a fixed fixed p_1, using Theorem 1 one gets:

$$E_{11}(p_1) = p_2'[p_1(a-b-c+d)+c-d] + p_1(b-d)+d$$
$$= p_1[p_2'(a-b-c+d)+b-d] + p_2'(c-d)+d \qquad (5)$$

Similarly regarding his second goal, $E_{12}(p_1)$ is obtained as:

$$E_{12}(p_1) = p_1[p_2'(e-f-g+h)+f-h] + p_2'(g-h)+h \qquad (6)$$

But since (4) and (5) are linear in p_1, so is the aggregated sum in (3), for a given weight w_1. Therefore the *slope* of the aggregated sum in (3) with respect to p_1 will determine the optimal strategy. Now, from (1), (2), (3), (5) and (6),

$$\frac{\partial}{\partial p_1}(w_1\mu_{11}(p_1)+w_2\mu_{12}(p_1))=w_1D_1+w_2D_2=D(\text{say}), \tag{7}$$

where

$$D_1=\left(p_2'(a-c)+(1-p_2')(b-d)\right)/(\max 1-\min 1) \tag{8}$$

and

$$D_2=\left(p_2'(e-g)+(1-p_2')(f-h)\right)/(\max 2-\min 2) \tag{9}$$

or

$$D_2=\left(p_2'(e-g)+(1-p_2')(f-h)\right)/(\min 2-\max 2) \tag{10}$$

accordingly, the player being cooperative or non-cooperative. Now if $D > 0$, the value of the aggregated sum in (3) increases as p_1 increases in [0, 1], leads to the conclusion that $p_1 = 1$ is the optimal strategy. Similarly, $D < 0$ implies that the optimal strategy is $p_1 = 0$. If $D = 0$, all the strategies are equally good. Now if the payoff values and the weights known the evaluation of D involves only the computation of p_2' which is simple for most of the fuzzy variables that usually arise in practice [3].

1 Conclusion

The general case of the fuzzy approach to 2×2 games has been solved using the idea of the expected value of a fuzzy variable. The method requires only the computation of the expected value of a fuzzy variable, in order to achieve the optimal strategies in the game.

References

[1] D. Butnariu: *Fuzzy Games: A Description of the Concept,* Fuzzy Sets and Systems, vol.**1**, pp.181-192, 1978.

[2] K.W. Li, F. Karray, K.W. Hipel, D.M. Kilgour: *Fuzzy Approaches to the Game of Chicken,* IEEE Transactions on Fuzzy Systems, vol.**9**, pp.608-623, 2001.

[3] B. Liu, Y.K. Liu: *Expected Value of a Fuzzy Variable and Fuzzy Expected Value Models,* IEEE Transactions on Fuzzy Systems, vol.**10**, pp.445-450, 2002.

[4] R. Mathew, M.R. Kaimal: *A Fuzzy Approach to Strategic Games Using Fuzzy Expected Value Models,* Journal of the CSI, vol.**34**, pp.40-47, 2004.

[5] Q. Song, A. Kandel: *A Fuzzy Approach to Strategic Games,* IEEE Transactions on Fuzzy Systems, vol.**7**, pp.634-642, 1999.