F.S.A.I., Vol. 10, Nos. 1–2, pp. 57– 65, 2004

Fuzzy Approach for Multiple Criteria Fractional Optimisation – Comments and Quadratic Constraint Generalization

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Abstract: In this paper a fallacy of the fuzzy solution procedure for multiple criteria linear fractional programming problem (MCLFPP) described by Dutta, Tiwari and Rao (1992) is noted. Moreover, new thresholds for nominators and denominators of criteria are defined in order to solve MCLFPP using Luhandjula's (1984) membership functions of linguistic variables. Some comments on a simple extension of the method to the case of multiple criteria nonlinear fractional programming problem are included. Including above remarks, the Dutta's fuzzy method is developed to solving multiple objective linear fractional optimization with quadratic constraint. Computational results are presented to illustrate the new method.

1 Introduction

Luhandjula [1] used a linguistic variable approach in order to present a procedure for solving MCLFPP. Dutta, Tiwari and Rao [2,3] modified the linguistic approach of Luhandjula such to obtain efficient solution to problem MCLFPP. Ravi and Reddy [4], instead of "Simple Additive Weighting " model of MCLFPP, replaced the addition operator by the "min" operator as a fuzzy aggregator. Stancu-Minasian and Pop [5] pointed out certain shortcomings in the work of Dutta *et al.* and given the correct proof of theorem which validates the obtaining of the efficient solutions for MCLFPP.

In Section 2 a fallacy of the fuzzy solution procedure for MCLFPP described by Dutta *et al.* is noted. New thresholds for nominators and denominators of criteria are defined in order to solve MCLFPP using Luhandjula's membership functions [1] of linguistic variables [6]. Numerical examples are also considered for illustration.

In Section 3, including the changes claimed by the results of Section 2, Dutta's fuzzy method is developed for solving MCLFPP with quadratic constraint. In Section 4 a multiple objective linear fractional programming problem with quadratic constraint is solved using the developed method. Also, Example 3.2 from [2] is analysed using Dutta’s approach in terms of modified thresholds. Concluding remarks are included in Section 5.

Part of this paper was presented in [7]. As fundaments of described method, more details and computational results are included here.

2 Dutta's fuzzy method to solving MCLFPP

Luhandjula proves in [1] the following lemma:

Lemma 2.1 Consider $a, b, c, d, \delta_1, \delta_2 \in \mathrm{R}$, $\delta_1 > 0$, $\delta_2 > 0$ and

$$V_x(\varepsilon)=\{y\in R \mid |y-x|<\varepsilon\}.$$

If $a\in V_b(\delta_1)$ and $c\in V_d(\delta_2)$ then

$$\frac{a}{c}\in V_{\frac{b}{d}}(\delta_3),\ \ \delta_3=\frac{\delta_1\delta_2}{|cd|}+\frac{|b|\delta_2+|c|\delta_1}{|cd|}.$$

Luhandjula assumes that if $|cd|$ is sufficiently large, then for small δ_1 and δ_2, δ_3 is sufficiently small.

Remark 2.1 Generally it is not true that the more *a* is close to *b* and *c* close to *d*, the more $\frac{a}{c}$ is sufficiently close to $\frac{b}{d}$.

Example 2.1 Let us consider the linear fractional function

$$f(x)=\frac{x_1+x_2+1}{x_1-x_2+2},\quad x\in[1,2]\times[1,2].$$

Putting

$$a(x)=x_1+x_2+1,\ c(x)=x_1-x_2+2,$$

evaluating

$$b=\max\{x_1+x_2+1 \mid x\in[1,2]\times[1,2]\}=5,$$

$$d=\min\{x_1-x_2+2 \mid x\in[1,2]\times[1,2]\}=1,$$

$$\max\left\{\frac{x_1+x_2+1}{x_1-x_2+2} \mid x\in[1,2]\times[1,2]\right\}=4$$

it follows that

$$\left|\frac{a(x)}{c(x)}-\frac{b}{d}\right|\geq 1, \forall x\in[1,2]\times[1,2].$$

Consequently, even $a(x)$ could be placed close to *b* and $c(x)$ could be placed close to *d*, ther is a gap between $\frac{a(x)}{c(x)}$ and $\frac{b}{d}$. Hence, $\frac{a(x)}{c(x)}$ could not be placed sufficiently close to $\frac{b}{d}$.

Example 2.1 also proves the validity of Remark 2.1 and the fallacy of Luhandjula's assumption. Starting from Luhandjula's assumption, Dutta *et al.* complete in [2] two other lemmas which permit to define the goals membership functions beginning with the concept of *(Z,ε)*-proximity used in the larger framework of the linguistic variables domain.

In their solving method, Dutta *et al.* considered the following membership functions:

$$C^N(x) = \begin{cases} 0, & N(x) < p \\ \dfrac{N(x) - p}{N^0 - p}, & p \le N(x) \le N^0 \\ 0, & N(x) > N^0 \end{cases} \tag{1}$$

$$C^D(x) = \begin{cases} 0, & D(x) > s \\ \dfrac{s - D(x)}{s - D^0}, & D^0 \le D(x) \le s \\ 0, & D(x) < D^0 \end{cases} \tag{2}$$

where N^0 and D^0 represent the maximal value of nominator *N(x)* and the minimal value of denominator *D(x)* on the feasible set of criteria

$$f(x) = \frac{N(x)}{D(x)},$$

while *p* and *s* are the thresholds beginning with which values nominator *N(x)* and *D(x)* are acceptable.

As membership function of a goal induced by the criterion *f(x)* it is chosen the function

$$\mu(x) = wC^N(x) + w'C^D(x)$$

where *w* and *w'* are the weights indicating the relative importance given by the decision maker to the criterion.

In what follows it is assumed that the membership function μ verifies the hypothesis

$$\forall x^1, x^2 \in X, \quad \text{if} \quad \frac{N(x^1)}{D(x^1)} > \frac{N(x^2)}{D(x^2)} \quad \text{then} \quad \mu(x^1) > \mu(x^2). \tag{3}$$

In [5] the authors defined equivalent descriptions for (3) when a concrete function *f(x)* is considered.

Due to Remark 2.1, a change of the values N^0 and D^0 has to be made in order to keep the same membership functions C^N, C^D and μ.

Considering the fractional programming problem

$$\max\left\{ f(x) = \frac{N(x)}{D(x)} \mid x \in X \right\}$$

and intending to aggregate it with other goals using the membership functions (1)-(2) the following definitions have to be used:

$$f(x^0) = \max\{f(x) \mid x \in X\},$$

$$N^0 = N(x^0), D^0 = D(x^0).$$

Hence, thresholds *p* and *s* will be established in concordance with new values N^0 and D^0 respectively.

Definitions which were considered above could be used even $N(x)$ and $D(x)$ are linear functions or not. When $N(x)$ and $D(x)$ are nonlinear functions some difficulties could appear in the further solving method: membership functions $C^N(x)$ and $C^D(x)$ described by (1)-(2) become nonlinear and Problem [P] in Section 3 will have linear objective function and nonlinear constraint.

It is clear that we formulated a counter-example to obtain conclusions which have also given solving solutions for the initial problem.

3 The fuzzy method to solving MCLFPP with quadratic constraints

Let us consider the multiple objective linear fractional programming problem with quadratic constraint:

$$\text{"max"}\left\{ f(x) = \left(\frac{N_1(x)}{D_1(x)}, \frac{N_2(x)}{D_2(x)}, \ldots, \frac{N_p(x)}{D_p(x)} \right) \mid x \in X \right\} \tag{4}$$

where

$$X = \left\{ x \in R^n \mid Ax \leq b,\, p'x + \frac{1}{2} x' Hx \leq \gamma,\, x \geq 0 \right\}$$ is a bounded set,

A is an $m \times n$ constraint matrix, x is an n-dimensional vector of decision variable and $b \in R^m$,

$$N_i(x) = (c^i)' x + d_i,\quad D_i(x) = (e^i)' x + f_i,\quad \forall i = \overline{1, p},$$
$$c^i, e^i \in R^n,\ d_i, f_i \in R,\ \forall i = \overline{1, p},$$
$$(e^i)' x + f_i > 0,\ \forall i = \overline{1, p},\ \forall x \in X.$$

The term "max" is used in Problem (4) for finding efficient solutions in a maximization sense in terms of classical definitions [8,9].

In [8] a solving method for linear fractional programming problem with quadratic constraint is described. We briefly present this method in terms of the problem

$$[P_i]: \quad \max\left\{ f_i(x) = \frac{N_i(x)}{D_i(x)} \mid x \in X \right\}.$$

A solution x_i^{opt} for $[P_i]$ is obtaining following procedure's LFQC($[P_i]$) steps:

Step 1: Define

$$Y = \{ x \in R^n \mid Ax \leq b, x \geq 0 \}.$$

Step 2: Optimize $f_i(x)$ on Y in order to obtain

$$\mu_0^i = f_i(x_*^i) = \max\{ f_i(x) \mid x \in Y \}.$$

Step 3: If

$$p' x_*^i + \frac{1}{2}\left(x_*^i\right)' H x_*^i \le \gamma$$

then stop with $x_i^{opt} = x_*^i$. Otherwise go to Step 4.

Step 4: Solve the parametric problem

$$\min\left\{ p' x + \frac{1}{2} x' H x \mid x \in Y, N_i(x) \ge \mu D_i(x) \right\}$$

starting with $\mu = \mu_0^i$ and finishing when γ is obtained as the value of the objective function.

This solving method for Problem [P_i] combines a linear fractional optimization in Step 2 with a parametric quadratic optimization in Step 4.

Finally, in this section we apply above mentioned method to solve MCLFPP with quadratic constraint using linguistic variables approach to aggregate multiple criteria.

The solving algorithm for MCLFPP consists in the following steps:

Step 1: Call LFQC([P_i]) for all $i=1,2,\ldots,p$ in order to obtain $\left(x_i^{opt}\right)_{i=\overline{1,p}}$.

Step 2: Define $N_i^0 = N_i\left(x_i^{opt}\right)$, $D_i^0 = D_i\left(x_i^{opt}\right)$ for all $i = 1,2,\ldots,p$ and establish the thresholds p_i, s_i, $i=1,2,\ldots,p$ close to N_i^0 and D_i^0 respectively.

Step 3: Construct the membership functions C^{N_i}, C^{D_i} for all $i = 1,2,\ldots,p$.

Step 4: Define the problem [P]

$$\max V(\mu) = \sum_{i=1}^{p}\left(w_i \mu_i^{N_i} + w'_i \mu_i^{D_i}\right)$$

subject to

$$\mu_i^{N_i} = C^{N_i}(x),\ \ \mu_i^{D_i} = C^{D_i}(x),$$

$$0 \le \mu_i^{N_i} \le 1,\ \ 0 \le \mu_i^{D_i} \le 1,\ \ i = \overline{1,p},$$

$$Ax \le b, x \ge 0,\ \ p' x + \frac{1}{2} x' H x \le \gamma,$$

$$\sum_{i=1}^{p}(w_i + w_i') = 1.$$

Step 5: Call LFQC([P]).
Applying this algorithm convenient solutions are obtained.

5 Computational results

In order to illustrate the developed method of solving multiple criteria linear fractional programming problem with quadratic constraint, let us consider the following example:

$$\text{"max"}\left(f_1(x) = \frac{x_1 + x_2 - 1}{-x_1 + 2x_2 + 7},\ f_2(x) = \frac{2x_1 + x_2 - 2}{x_2 + 4} \right) \quad (5)$$

subject to

$$\begin{aligned} &-x_1 + 3x_2 \le 0, \\ &x_1 \le 6, \\ &x_1, x_2 \ge 0, \end{aligned} \quad (6)$$

$$x_1^2 + x_2^2 - 6x_1 - 4x_2 \le -2 \quad (7)$$

Graphical representation of the feasible set (6)-(7) is described in Figure 1.

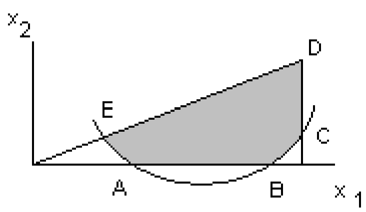

Figure 1.

The extremal points of the feasible set (6)-(7) are *A*(0.35,0), *B*(5.64,0), *C*(6,0.58), *D*(6,2), *E*(0.3,0.1).

i	1	2
x_*^i	(6,0)	(6,0)
$f_i\left(x_*^i\right)=\mu_0^i$	5	5/2
x_{opt}^i	(5.64,0)	(5.76,1.67)
$f_i\left(x_{opt}^i\right)$	3.41	2.32

Considering Problem (5)-(6)-(7) and running LFQC([P$_1$]) and LFQC([P$_2$]) the results grouped in table above are obtained. Using these results thresholds p_i and s_i are define in concordance with values N_i^0 and D_i^0 as follows:

	p_i	S_I	N_i^0	D_i^0
$i = 1$	7	1.35	4.64	1.36
$i = 2$	12	4.095	11.19	5.67

Using these results we define Problem [P] as follows:

$$\max\left\{V(\mu)=w_1\mu_1+w_2\mu_2+w_3\mu_3+w_4\mu_4\right\}$$

subject to

$$\begin{aligned}
&-x_1+3x_2\le 0, x_1\le 6, x_1, x_2\ge 0,\\
&x_1^2+x_2^2-6x_1-4x_2\le -2,\\
&C^{N_1}(x)=\mu_1, C^{D_1}(x)=\mu_2,\\
&C^{N_2}(x)=\mu_3, C^{D_2}(x)=\mu_4,\\
&0\le \mu_1,\mu_2,\mu_3,\mu_4\le 1,\\
&w_1+w_2+w_3+w_4=1.
\end{aligned}$$

Using Dutta's weights x = (5.65, 1.57) as solution of Problem (5)-(6)-(7) is obtained. Initial objective functions reached the values (2.935, 2.321) which are comparative acceptable with (3.41, 2.32).

We now consider Example 3.2 from [2] which is the same with Problem (5)-(6). Dutta's thresholds are p_i and s_i. Changing these thresholds in a_i and b_i respectively no feasible solution could be obtained. It means that the efficient point (6,0) of Problem (5)-(6) could not be reached despite it exists and it is unique.

	p_i	S_I	a_i	b_i
i = 1	*4*	*3*	*5.5*	*1*
i = 2	*8*	*7*	*10.5*	*4*

5 Concluding remarks

In this work a new method to solving multiple objective linear fractional programming problem with quadratic constraint was presented.

To develop this method, the solving algorithm for a single objective linear fractional programming problem with quadratic constraint and a fuzzy approach for multiple criteria linear fractional optimization were combined.

Starting from Dutta's fuzzy approach some changes on the thresholds of acceptable criteria's values were made in order to avoid a fallacy in the definitions of membership functions of the linguistic variables.

References

[1] M.K. LUHANDJULA: *Fuzzy Approaches for Multiple Objective Linear Fractional Optimization;* Fuzzy Sets and Systems, vol.13, p.11-23, 1984.

[2] D. DUTTA, R.N. TIWARI, J.R. RAO: *Multiple Objective Linear Fractional Programming Problem – A Fuzzy Set Theoretic Approach;* Fuzzy Sets and Systems, vol.52, p.39-45, 1992

[3] D. DUTTA, J.R. RAO, R.N. TIWARI: *Fuzzy Approaches for Multiple Criteria Linear Fractional Optimization: A Comment;* Fuzzy Sets and Systems, vol.54, p.347-349, 1993.

[4] V. RAVI, P.J. REDDY: *Fuzzy Linear Fractional Goal Programming Applied to Reffinery Operations Planning;* Fuzzy Sets and Systems, vol.962, p.173-182, 1998.

[5] I.M. STANCU-MINASIAN, B. POP: On a Fuzzy Set Approach to Solving Multiple Objective Linear Fractional Programming Problem; Fuzzy Sets and Systems vol.134, p.397-405, 2003.

[6] L.A. ZADEH: *The Concept of Linguistic Variable and its Application to Approximate Reasoning;* Information Sciences, vol.8, p.199-244, 1975.

[7] B. POP: *Fuzzy Set Approach for Multiple ObjectiveLinear Fractional Optimization with Quadratic Constraint;* Proceeding of ICCC 2004, Baile-Felix Spa-Oradea, Romania, p.355-358, 2004.

[8] I.M. STANCU-MINASIAN: *Fractional Programming: Theory, Methods and Applications;* ed. Kluwer Academic Publishers, Dordrecht, 1997.

[9] I.M. STANCU-MINASIAN: *Stochastic Programming with Multiple Objective Functions;* ed. D. Reidel Publishing Company, Dordrecht/Boston/Lancester, 1984.