F.S.A.I., Vol. 9, Nos. 1–3, pp. 59–88, 2003

A New Fuzzy Inference Method
Based on Spatial Relationship of Fuzzy Subsets

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Abstract: In this study, a new inference method, which matches the intuition of human reasoning is proposed. For each fuzzy rule "if *x* is $\tilde{A}$, then *y* is $\tilde{B}$," the proposed inference method produces an output fuzzy subset $\tilde{B}'$ according to the spatial relationship of input fuzzy subset $\tilde{A}'$ and the antecedent $\tilde{A}$. The membership function of the output fuzzy subset $\tilde{B}'$ is of the same shape as that of the input fuzzy subset $\tilde{A}'$. Moreover, the spatial relationship of $\tilde{B}$ and $\tilde{B}'$ is consistent with that of $\tilde{A}$ and $\tilde{A}'$. Experiments on a simple single-input single-output example and a truck backer-parking controller are performed to compare our proposed inference method with the Mamdani's method. These experiments show that our results compare favourably with those of the Mamdani's method.
Keywords: fuzzy inference, fuzzy logic, human reasoning, spatial relationship, fuzzy set

1. Introduction

Since the fuzzy set theory was proposed by Zadeh in 1965 [1], it has matured into a wide-ranging collection of concepts and techniques for dealing with complex phenomena which are not analyzed based on the classical probabilistic theory. Fuzzy logic has achieved successful performance and popularity in many areas such as control, decision making, approximate reasoning, pattern recognition, image processing and so on [2-4]. Fuzzy logic inference plays an important role between the human and machine. The application of fuzzy logic to practice has the advantage of greater simplicity, greater robustness, and lower cost in comparison with the traditional methods.

Many fuzzy logic inference methods have been proposed. One approach to fuzzy inference is based on the compositional rule of inference, proposed by Zadeh in 1973 [5]. The max-min composition is used in the rule inference. In the method proposed by Mamdani [6, 7], its fuzzy implication operator is the minimum operator. The input can be either a crisp value or a fuzzy subset and the output is a fuzzy subset. The output fuzzy subset is then converted to a crisp value by a defuzzifier. This method is simple and easy to implement. Another approach is the

TSK fuzzy inference method suggested by Takagi, Sugeno, and Kang [8-10], which is a multi-dimensional method based on a fuzzy partition of the input space.

For each fuzzy subspace, a linear relation of input-output is formed. And fuzzy rules are generated from a given input-output data set. The fuzzy rule is of the form: "if x_1 is $\tilde{A}_1$, x_2 is $\tilde{A}_2$, ..., x_k is $\tilde{A}_k$ then y=f(x_1, x_2, ..., x_k)," where the y function produces output based on the fuzzy subspace specified by the antecedent of the rule. The inference output is a crisp value and thus defuzzification is unnecessary. Another approach is proposed by Tsukamoto [11]. It is assumed that the consequent of a fuzzy rule is a fuzzy subset with a monotonical membership function. The inference output of each rule is a crisp value obtained from the rule's firing strength. The final inferred result is the weighted average of all the fired rules' output. Since the output is a crisp value, defuzzification is also unnecessary. Another inference method is based on neural networks [12, 13], which brings the learning abilities to fuzzy logic systems. The fuzzy neural network can be constructed automatically by learning the given training examples. This approach saves the rule-matching time spent in traditional fuzzy inference methods. There are many other fuzzy inference approaches proposed for different applications, which can be referred from the literature [14-16].

The inference results produced by the above mentioned methods do not match human intuition in general. For example, the same inference output may be obtained when different input fuzzy subsets are given to the Mamdani's method, as shown in Fig. 1. By intuition, the two different input fuzzy subsets should result in different output fuzzy subsets. In this study, we propose an inference method, which produces appropriate output fuzzy subsets according to the spatial relationship of related fuzzy subsets. In the method, the input is assumed to be a fuzzy subset. If it is not, it is converted to a fuzzy subset by a fuzzifier. The membership function of our output fuzzy subset is of the same shape as that of the input fuzzy subset. For instance, when both the antecedent and the consequent of a fuzzy rule are of triangular shape but the input is of trapezoid shape, the output will be of trapezoid shape. Experiments are performed on a single-input single-output example and a truck backer-parking example to compare our proposed inference method with the Mamdani's method. The advantages of our proposed method are that smoother input-output curve can be obtained and greater robustness can be achieved.

The remainder of this paper is organized as follows. The detail of the proposed approach is described in Section 2. In Section 3, experimental results using both the proposed inference method and the Mamdani's inference method are given. Conclusions appear in the last section.

Fuzzy rule: If x is $\tilde{A}$ then y is $\tilde{B}$.

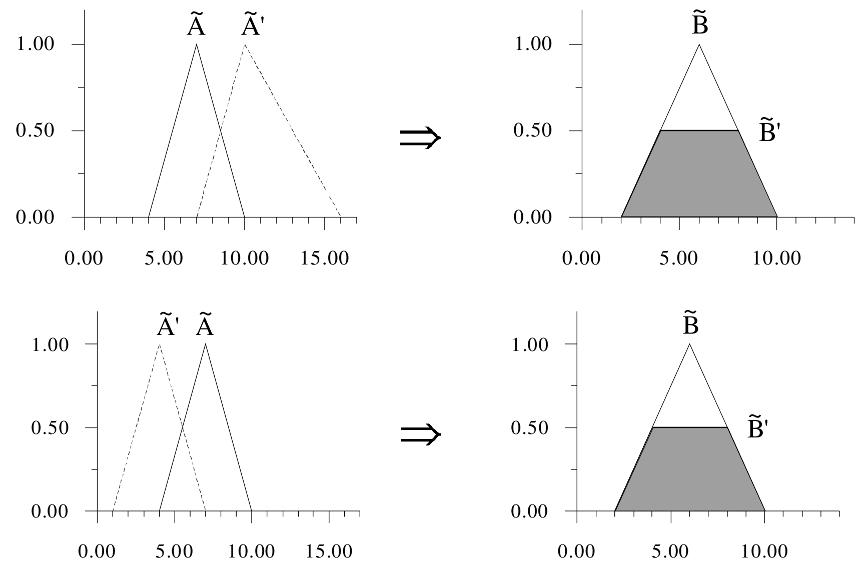


Fig. 1 – The same output fuzzy subset $\tilde{B}'$ is obtained when different input fuzzy subsets $\tilde{A}'$ are given to inference in the Mamdani's method.

2. Proposed Approach to Fuzzy Logic Inference

The block diagram of a typical fuzzy control system is shown in Fig. 2. The fuzzy logic inference includes, in general, three parts: fuzzifier, inference engine, and defuzzifier. In a fuzzy system, the membership functions and fuzzy rules are frequently defined by experts in advance. When crisp data is input, it is first fuzzified into either a singleton or a fuzzy subset by the fuzzifier. Then the fuzzified result is input to the inference engine and fuzzy-rule matching is performed. Finally, the inferred fuzzy subset is defuzzified into a scalar (crisp number) by the defuzzifier. The scalar is used to control the system plant and the system output is then inserted into the fuzzifier again.

Though many fuzzy logic inference approaches have been suggested, their results do not closely imitate the intuition of human reasoning. In this paper, we propose a new fuzzy logic inference method, which matches human reasoning. The proposed method is based on the spatial relationship of related fuzzy subsets to draw an inference. Let the fuzzy rule from which we make an inference be " If x is $\tilde{A}$ then y is $\tilde{B}$," where $\tilde{A}$ and $\tilde{B}$ are fuzzy subsets defined in the universes of discourse. Assume that the input fuzzy subset is $\tilde{A}'$. The output fuzzy subset $\tilde{B}'$ is then produced according to the spatial relationship of $\tilde{A}$ and $\tilde{A}'$. The spatial relationship of fuzzy subsets $\tilde{B}$ and $\tilde{B}'$ is similar to that of $\tilde{A}$ and $\tilde{A}'$.

To specify the spatial relationship of fuzzy subsets, each fuzzy rule in the fuzzy system should be given a sign mark a priori. This mark is used to determine the relationship of $\tilde{A}$ and $\tilde{B}$. For example, shown in Fig. 3 are two fuzzy rules: "If someone takes exercise, then he wears off the fat," and "If someone takes exercise, then he exudes perspiration." The inference results of these two rules are quite different for the case that someone takes more exercise, as shown in Fig. 3 When someone takes more exercise, he wears off more fat and thus his fat is reduced. Hence the output fuzzy subset $\tilde{B}'$ of the first rule is on the left of $\tilde{B}$, as Fig. 3(a) shows. In this case, however, he exudes more perspiration. Therefore, the output fuzzy subset $\tilde{B}'$ of the second rule is on the right of $\tilde{B}$, as Fig. 3(b) shows. It is reasonable to discriminate these two situations in fuzzy inference. But it is neglected in most fuzzy inference methods. In this study, a negative sign mark denotes the first situation (...more..., ...less...), and a positive sign mark denotes the second situation (...more..., ...more...). By the sign mark, the output fuzzy subset can be appropriately located.

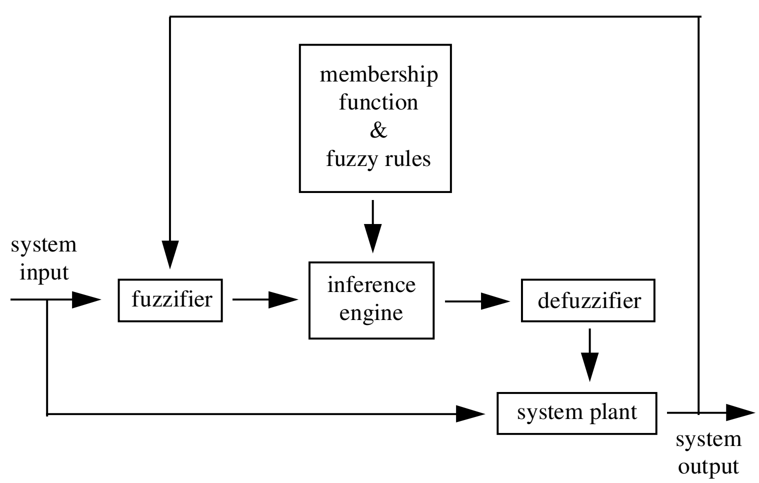


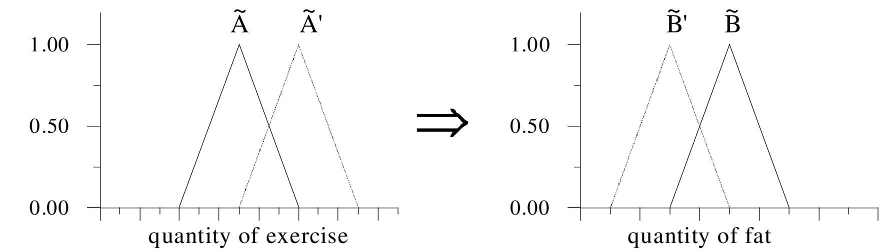
Fig. 2 – The block diagram of a typical fuzzy control system.

By intuition, if the antecedent $\tilde{A}$, the consequent $\tilde{B}$, and the input $\tilde{A}'$ are all of triangular shape, then the output $\tilde{B}'$ should be triangular. But when $\tilde{A}$, $\tilde{B}$ and $\tilde{A}'$ are all of trapezoid shape, the output $\tilde{B}'$ should be a trapezoid. Hence in the proposed inference method, given the shape of membership functions for $\tilde{A}$, $\tilde{B}$ and $\tilde{A}'$, we derive the same shape of membership function for the output $\tilde{B}'$. Membership functions of triangular, trapezoid and Π shapes are frequently used in many applications [4]. Therefore the membership functions for $\tilde{A}$, $\tilde{B}$, and $\tilde{A}'$ are assumed to be of these shapes in this study. Four or less parameters are enough to represent a membership function of the three shapes. For example, four parameters suffice for representing a membership function of trapezoid shape, as shown in Fig.

4(a). Triangulars are special cases of trapezoids. When the parameter *c* is equal to the parameter *d* in a trapezoid, it becomes a triangular, as shown in Fig. 4 (b).

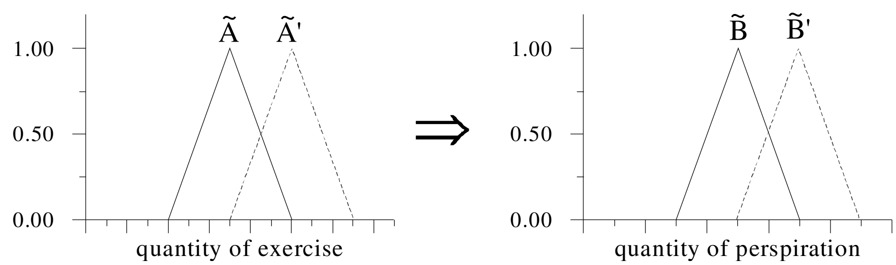
To specify a membership function of Π shape, only two parameters are enough, as shown in Fig. 4(c).

Fuzzy rule: If someone takes exercise, then he wears off the fat.



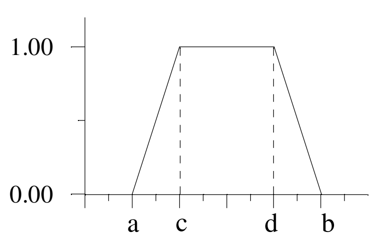
(a)

Fuzzy rule: If someone takes exercise, then he exudes perspiration.



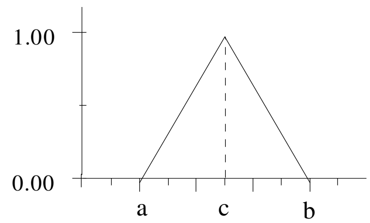
(b)

Fig. 3 The inferred results $\tilde{B}'$ of two fuzzy rules.



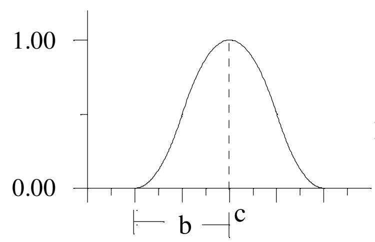
$$y = \begin{cases} \frac{1}{c-a}(x-a) & a \le x \le c \\ 1 & c \le x \le d \\ \frac{1}{d-b}(x-b) & d \le x \le b \end{cases}$$

(a) Membership function of trapezoid shape.



$$y=\begin{cases}\dfrac{1}{c-a}(x-a) & a\le x\le c\\ \dfrac{1}{c-b}(x-b) & c\le x\le b\end{cases}$$

(b) Membership function of triangular shape.



$$\Pi(x;b,c)=\begin{cases}2\left(\dfrac{x-c+b}{b}\right)^2 & c-b\le x\le c-\dfrac{b}{2}\\ 1-2\left(\dfrac{x-c}{b}\right)^2 & c-\dfrac{b}{2}\le x\le c+\dfrac{b}{2}\\ 2\left(\dfrac{x-c-b}{b}\right)^2 & c+\dfrac{b}{2}\le x\le c+b\end{cases}$$

(c) Membership function of Π shape.

Fig. 4 – Three frequently used membership functions.

2.1. Derivation of Membership Functions of Trapezoid and Triangular Shapes

In this section, we first consider the derivation of $\tilde{B}'$ whose membership function is of trapezoid shape (i.e., the membership functions of $\tilde{A}$, $\tilde{B}$, and $\tilde{A}'$ are all of trapezoid shape). Four parameters are enough to represent a trapezoid membership function. Hence four equations are required for solving the four parameters of $\tilde{B}'$.

Before deriving the four equations, we introduce four features, which characterize the shape and location of a trapezoid fuzzy subset. They are defined as follows:

$$t=\frac{b+d-a-c}{2} \tag{1}$$

$$u=\frac{\int_x xf(x)dx}{\int_x f(x)dx}=\frac{(b^2+d^2-a^2-c^2+b\times d-a\times c)}{3\times(b+d-a-c)} \tag{2}$$

$$v=\frac{c+d}{2}-a \tag{3}$$

$$w = b - \frac{c+d}{2} \tag{4}$$

where *f*(*x*) denotes the membership function of a trapezoid fuzzy subset, and *a, b, c,* and *d* are four parameters of the trapezoid fuzzy subset, as shown in Fig. 4(a). Feature *t* denotes the area of a trapezoid, and feature *u* represents the center of gravity of a trapezoid. Features *v* and *w* denote the lengths from *a* to (*c*+*d*)/2 and from (*c*+*d*)/2 to *b* of a trapezoid, respectively.

The above four features are computed for each of the fuzzy subsets $\tilde{A}$, $\tilde{B}$, and $\tilde{A}'$ before $\tilde{B}'$ is derived. The four features of $\tilde{A}$ are denoted as $t_{\tilde{A}}$, $u_{\tilde{A}}$, $v_{\tilde{A}}$ and $w_{\tilde{A}}$. The similar notations are used for the features of $\tilde{B}$ and $\tilde{A}'$. In the proposed inference method, the output fuzzy subset $\tilde{B}'$ is derived according to the features of $\tilde{A}$, $\tilde{B}$ and $\tilde{A}'$. We obtain the four parameters of $\tilde{B}'$ by solving the following four equations:

$$\frac{t_{\tilde{B}}}{t_{\tilde{A}}} = \frac{t_{\tilde{B}}}{t_{\tilde{A}}} \tag{5}$$

$$\pm\left(\frac{u_{\tilde{B}} - u_{\tilde{B}}}{u_{\tilde{A}} - u_{\tilde{A}}}\right) = \frac{t_{\tilde{B}}}{t_{\tilde{A}}} \tag{6}$$

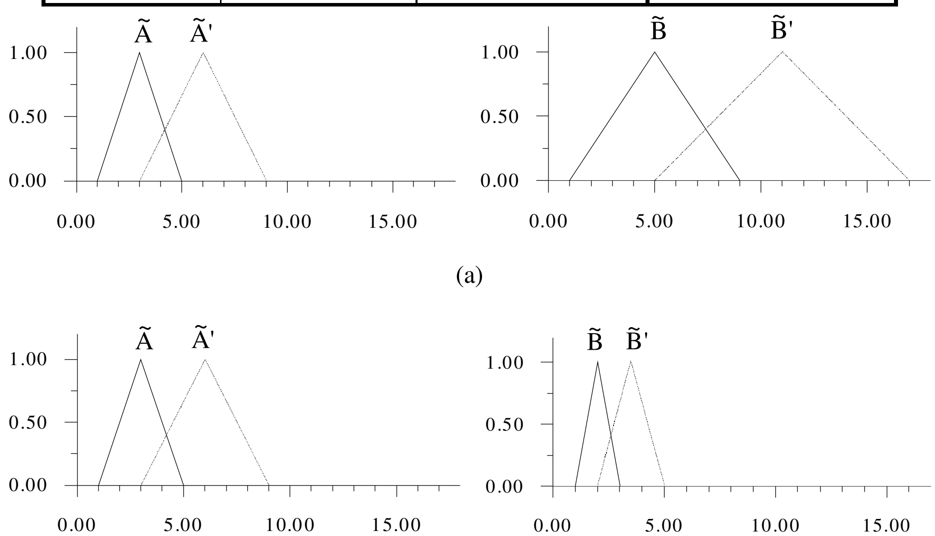
$$\frac{v_{\tilde{B}}}{v_{\tilde{A}}} = \frac{v_{\tilde{B}}}{v_{\tilde{A}}} \tag{7}$$

$$\frac{w_{\tilde{B}}}{w_{\tilde{A}}} = \frac{w_{\tilde{B}}}{w_{\tilde{A}}} \tag{8}$$

To derive $\tilde{B}'$, we assume in Equation (5) that the ratio between areas of fuzzy subsets $\tilde{B}$ and $\tilde{A}$ is equal to that of $\tilde{B}'$ and $\tilde{A}'$. In Equation (6), the sign mark of the left term is determined according to the sign mark of the fuzzy rule "If *x* is $\tilde{A}$, then *y* is $\tilde{B}$.". The value of ($u_{\tilde{B}'}$-$u_{\tilde{B}}$) is the distance between gravity centers of $\tilde{B}'$ and $\tilde{B}$, and the value of ($u_{\tilde{A}'}$-$u_{\tilde{A}}$) is that of $\tilde{A}'$ and $\tilde{A}$. The location of $\tilde{B}'$ with respect to $\tilde{B}$ (i.e., $\tilde{B}'$ is either on the left or on the right of $\tilde{B}$) can be determined according to Table 1. It is clear that if the area of $\tilde{B}$ is larger, the distance between gravity centers of $\tilde{B}'$ and $\tilde{B}$ is also larger. Two examples for illustration are given in Fig. 5. Hence the ratio between ($u_{\tilde{B}'}$-$u_{\tilde{B}}$) and ($u_{\tilde{A}'}$-$u_{\tilde{A}}$) is set equal to the ratio between areas of $\tilde{B}$ and $\tilde{A}$ in Equation (6). By the equation, the location relation of $\tilde{A}'$ and $\tilde{A}$ is preserved in that of $\tilde{B}'$ and $\tilde{B}$. In Equation (7), the ratio between lengths from *a* to (*c*+*d*)/2 in $\tilde{B}'$ and $\tilde{A}'$ is assumed equal to that in $\tilde{B}$ and $\tilde{A}$. In Equation (8), the ratio between lengths from $(c+d)/2$ to b in $\tilde{B}'$ and $\tilde{A}'$ is set equal to that in $\tilde{B}$ and $\tilde{A}$. By Equations (5)-(8), we can obtain $\tilde{B}'$ such that the spatial relationship of $\tilde{B}'$ and $\tilde{B}$ is similar to that of $\tilde{A}'$ and $\tilde{A}$.

Table 1 The location relation of $\tilde{B}$ and $\tilde{B}'$.

Sign mark of rule	sign of ($u_{\tilde{A}'}$-$u_{\tilde{A}}$)	sign of ($u_{\tilde{B}'}$-$u_{\tilde{B}}$)	location of $\tilde{B}'$ w.r.t. $\tilde{B}$
+	positive	positive	right
+	negative	negative	left
-	negative	positive	right
-	positive	negative	left



(b)

Fig. 5 – Examples to show that if the area of $\tilde{B}'$ is larger, the distance between gravity centers of $\tilde{B}'$ and $\tilde{B}$ is also larger.

The membership function of triangular shape can be considered as a special case of that of trapezoid shape. When the two parameters, c and d of a trapezoid are equal, the trapezoid becomes a triangular. Derivation of the four parameters for a trapezoid by Equations (5)-(8) is also applicable for a triangular. Fig. 6 shows an example of inference for membership functions of triangular shape. Fig. 7 shows an example of inference for membership functions of trapezoid shape. Figs. 8 and 9 show examples of inference for membership functions of either triangular or trapezoid shape. We know from the examples that when $\tilde{A}$ and $\tilde{B}$ are of triangular (trapezoid) shape and $\tilde{A}'$ is a trapezoid (triangular), the output $\tilde{B}'$ is a trapezoid (triangular). These results are consistent with the intuition of human reasoning.

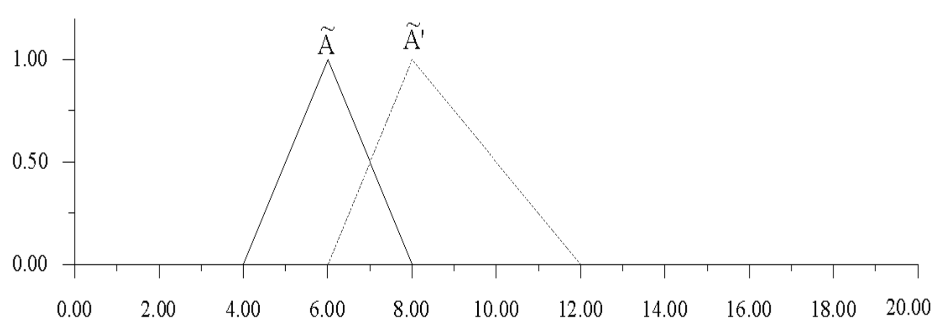
2.2. Derivation of Membership Functions of Π Shape

When the membership function of $\tilde{B}'$ is of Π shape, only two parameters *b* and *c*, as shown in Fig. 4(c), need to be determined. There are two features defined for the Π shape. One feature is its area, which is equal to the value of the bandwidth *b*. And the other feature is the center *c* of Π shape. We denote the two features for $\tilde{A}$ as $b_{\tilde{A}}$ and $c_{\tilde{A}}$. The similar notations are used for the features of $\tilde{B}$ and $\tilde{A}'$. In the proposed fuzzy logic inference method, the two parameters *b* and *c* of $\tilde{B}'$ are derived based on the features of $\tilde{A}$, $\tilde{B}$, and $\tilde{A}'$. They are obtained by solving the following two equations:

$$\frac{b_{\tilde{B}}}{b_{\tilde{A}}} = \frac{b_{\tilde{B}}}{b_{\tilde{A}}} \tag{9}$$

$$\pm\left(\frac{c_{\tilde{B}} - c_{\tilde{B}}}{c_{\tilde{A}} - c_{\tilde{A}}}\right) = \frac{b_{\tilde{B}}}{b_{\tilde{A}}} \tag{10}$$

In Equation (9), we assume that the ratio between areas of $\tilde{B}$ and $\tilde{A}$ equals that of $\tilde{B}'$ and $\tilde{A}'$. In equation (10), the sign mark of the left term is also determined from the sign mark of the fuzzy rule. In this equation, ($c_{\tilde{B}'}$-$c_{\tilde{B}}$) denotes the distance between centers of $\tilde{B}'$ and $\tilde{B}$ and ($c_{\tilde{A}'}$-$c_{\tilde{A}}$) denotes that of $\tilde{A}'$ and $\tilde{A}$. The location of $\tilde{B}'$ with respect to $\tilde{B}$ can also be determined according to Table 1. The larger area of $\tilde{B}$ compared with the area of $\tilde{A}$ results in the larger distance between centers of $\tilde{B}'$ and $\tilde{B}$. The ratio of ($c_{\tilde{B}'}$-$c_{\tilde{B}}$) to ($c_{\tilde{A}'}$-$c_{\tilde{A}}$) is thus set equal to the ratio of areas for $\tilde{B}$ and $\tilde{A}$. This equation preserves the location relation of $\tilde{A}'$ and $\tilde{A}$ in that of $\tilde{B}'$ and $\tilde{B}$. Fig. 10 shows an example of inference for membership functions of Π shape.



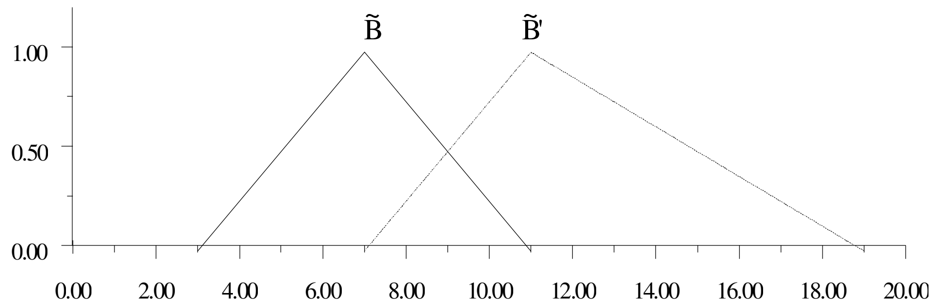


Fig. 6 – Inference for membership functions of triangular shape.

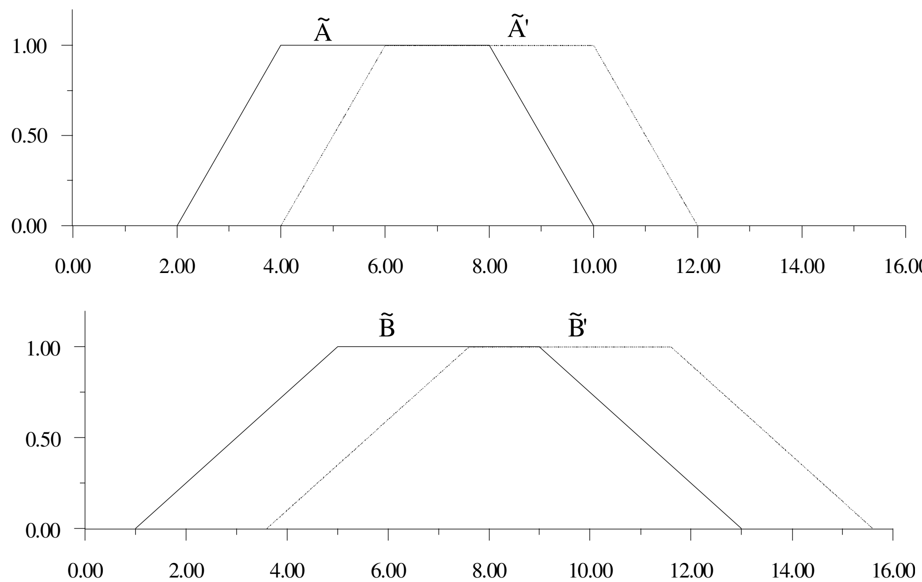
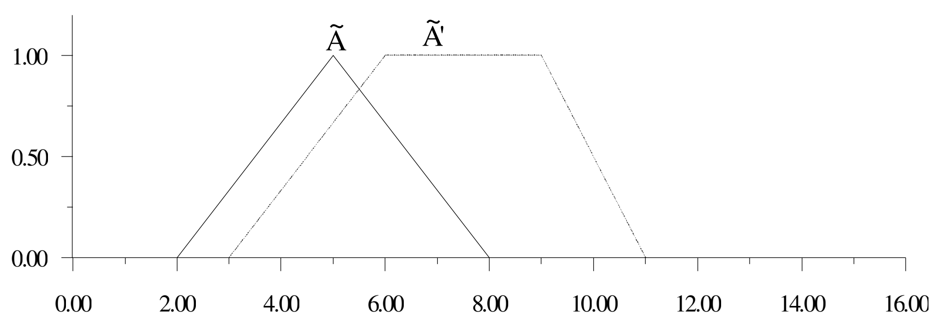


Fig. 7 – Inference for membership functions of trapezoid shape.



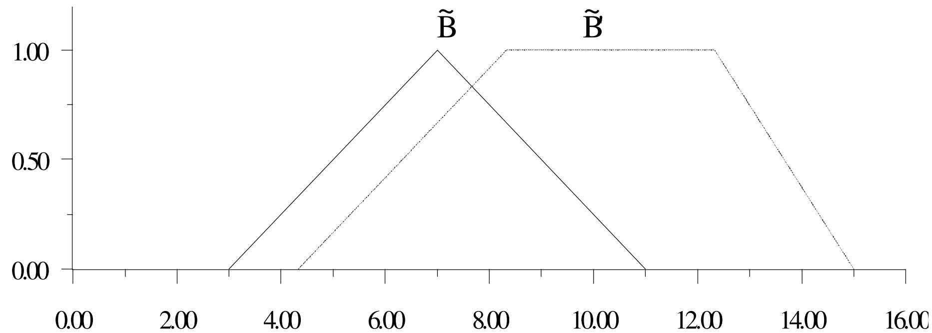


Fig. 8 – Inference for the case that $\tilde{A}$ and $\tilde{B}$ are of triangular shape and $\tilde{A}'$ is of trapezoid shape.

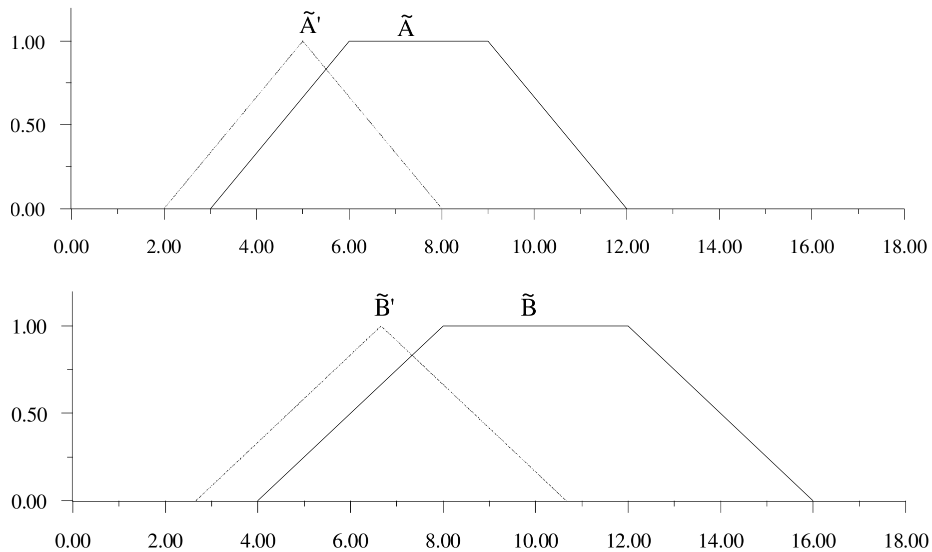
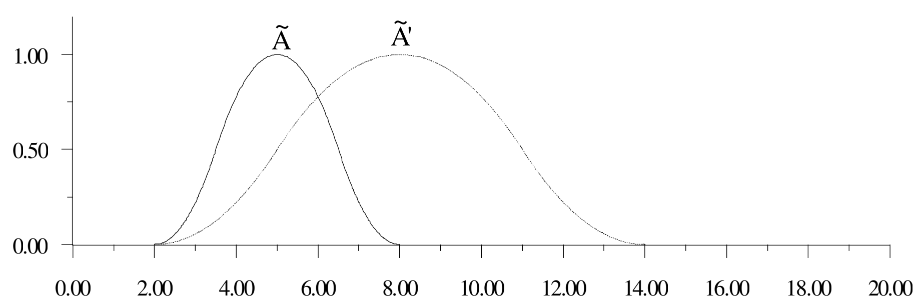


Fig. 9 – Inference for the case that $\tilde{A}$ and $\tilde{B}$ are of trapezoid shape and $\tilde{A}'$ is of triangular shape.



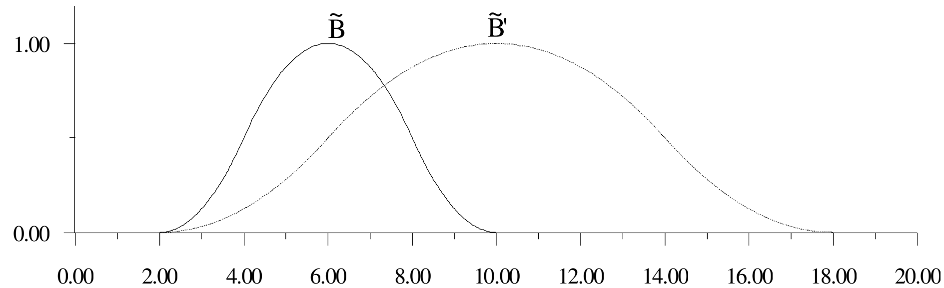


Fig. 10 – Inference for membership functions of Π – shape.

2.3. Discussion of Inference Criteria Satisfaction

In fuzzy logic inference, four basic criteria are frequently discussed [17, 18]:

Criterion 1 : IF x is $\tilde{A}$ THEN y is $\tilde{B}$
x is $\tilde{A}$

---

y is $\tilde{B}$

Criterion 2 : IF x is $\tilde{A}$ THEN y is $\tilde{B}$
x is very $\tilde{A}$

---

y is very $\tilde{B}$

Criterion 3 : IF x is $\tilde{A}$ THEN y is $\tilde{B}$
x is more or less $\tilde{A}$

---

y is more or less $\tilde{B}$

Criterion 4(I) : IF x is $\tilde{A}$ THEN y is $\tilde{B}$
x is not $\tilde{A}$

---

y is unknown

Criterion 4(II) : IF x is $\tilde{A}$ THEN y is $\tilde{B}$
x is not $\tilde{A}$

---

y is not $\tilde{B}$

The Zadeh's method [5] only satisfies Criterion 4(I). The Mamdani's method [6] only satisfies Criterion 1. The Mizumoto's method [19] satisfies Criteria 1, 2, 3 and 4(I). When the membership functions are of isosceles-triangular shape or Π shape, our inference method satisfies Criteria 1, 2 and 3. For Criterion 4, if we know in advance that the membership function of $\tilde{A}'$ is the result of a "not" operation on a membership function, the proposed inference method satisfies Criterion 4(II). Fig. 11 shows four examples of fuzzy logic inference by the proposed method, which illustrate the satisfaction of the four criteria. A simple proof for membership functions of isosceles- triangular shape is given below.

An isosceles triangular can be characterized by two parameters *c* and *h*, as shown in Fig. 12. Assume that the fired rule is "If *x* is $\tilde{A}$ then *y* is $\tilde{B}$." Let the membership functions of $\tilde{A}$ and $\tilde{B}$ be of isosceles-triangular shape. Also let the input fuzzy subset $\tilde{A}'$ be of isosceles-triangular shape. Two parameters *c* and *h* of $\tilde{A}$ are represented as $c_{\tilde{A}}$ and $h_{\tilde{A}}$, respectively. The similar notations are used to represent the parameters of $\tilde{B}$, $\tilde{A}'$, and $\tilde{B}'$. By Equations (1)-(4), the four features for each of $\tilde{A}$, $\tilde{B}$, and $\tilde{A}'$ are computed as follows:

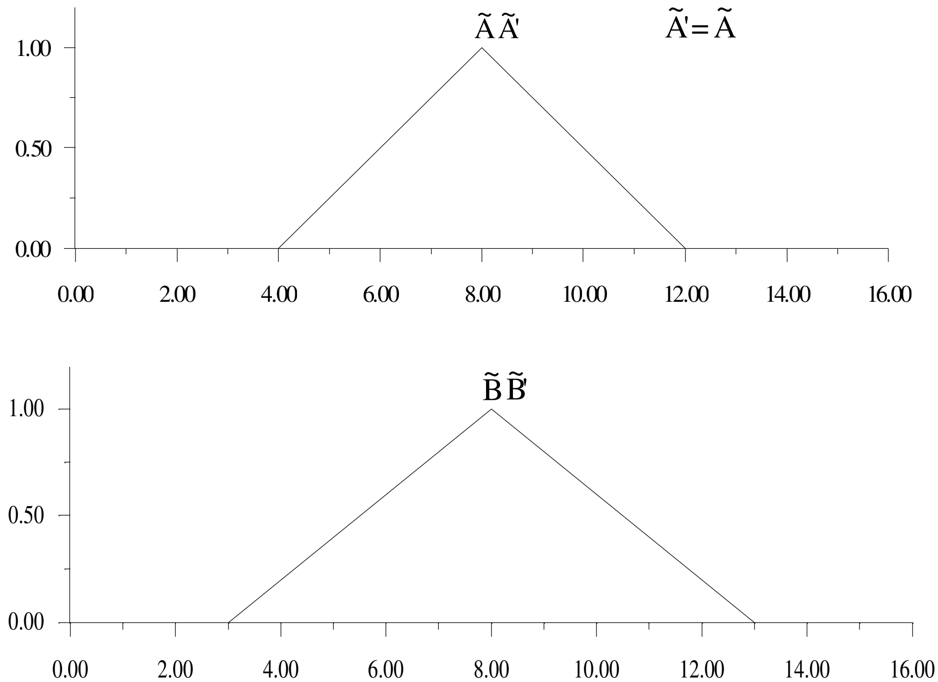
$$\begin{cases} t_{\tilde{A}} = \dfrac{h_{\tilde{A}}}{2} \\ u_{\tilde{A}} = 4 \times c_{\tilde{A}} \\ v_{\tilde{A}} = h_{\tilde{A}} \\ w_{\tilde{A}} = h_{\tilde{A}} \end{cases} \qquad \begin{cases} t_{\tilde{B}} = \dfrac{h_{\tilde{B}}}{2} \\ u_{\tilde{B}} = 4 \times c_{\tilde{B}} \\ v_{\tilde{B}} = h_{\tilde{B}} \\ w_{\tilde{B}} = h_{\tilde{B}} \end{cases} \qquad \begin{cases} t_{\tilde{A}} = \dfrac{h_{\tilde{A}}}{2} = \dfrac{h_{\tilde{A}}}{2} \\ u_{\tilde{A}} = 4 \times c_{\tilde{A}} = 4 \times c_{\tilde{A}} \\ v_{\tilde{A}} = h_{\tilde{A}} = h_{\tilde{A}} \\ w_{\tilde{A}} = h_{\tilde{A}} = h_{\tilde{A}} \end{cases}$$

(1) proof of Criterion 1:

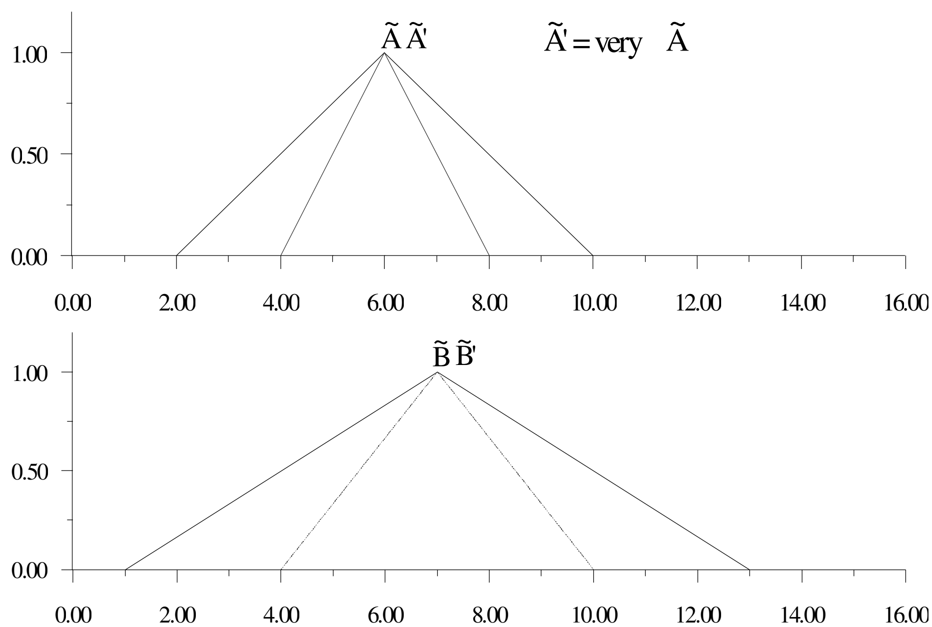
Assume that the input fuzzy subset $\tilde{A}' = \tilde{A}$. Using Equations (5)-(8), we have

$$\begin{cases} t_{\tilde{B}} = t_{\tilde{A}} \times \dfrac{t_{\tilde{B}}}{t_{\tilde{A}}} = \dfrac{h_{\tilde{B}}}{2} \\ u_{\tilde{B}} = u_{\tilde{B}} + \dfrac{t_{\tilde{B}}}{t_{\tilde{A}}}(u_{\tilde{A}} - u_{\tilde{A}}) = 4 \times c_{\tilde{B}} \\ v_{\tilde{B}} = v_{\tilde{A}} \times \dfrac{v_{\tilde{B}}}{v_{\tilde{A}}} = h_{\tilde{B}} \\ w_{\tilde{B}} = w_{\tilde{A}} \times \dfrac{w_{\tilde{B}}}{w_{\tilde{A}}} = h_{\tilde{B}} \end{cases} \Rightarrow \begin{cases} c_{\tilde{B}} = c_{\tilde{B}} \\ h_{\tilde{B}} = h_{\tilde{B}} \end{cases}$$

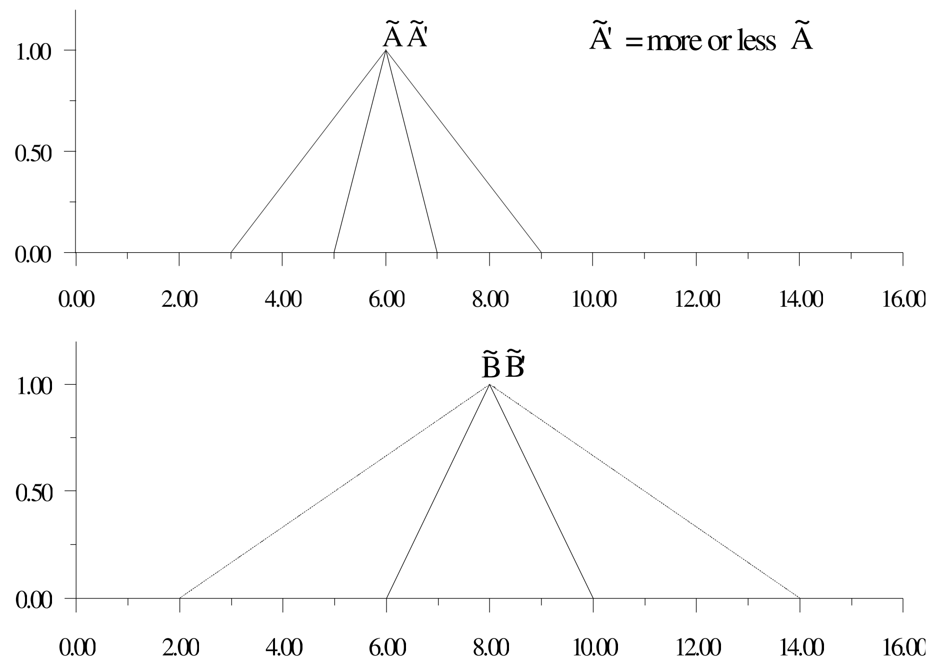
Hence, the output fuzzy subset $\tilde{B}'$ is indeed the same as $\tilde{B}$.


$\tilde{A}\ \tilde{A}'$
$\tilde{A}' = \tilde{A}$
1.00
0.50
0.00
0.00 2.00 4.00 6.00 8.00 10.00 12.00 14.00 16.00
$\tilde{B}\ \tilde{B}'$
1.00
0.50
0.00
0.00 2.00 4.00 6.00 8.00 10.00 12.00 14.00 16.00

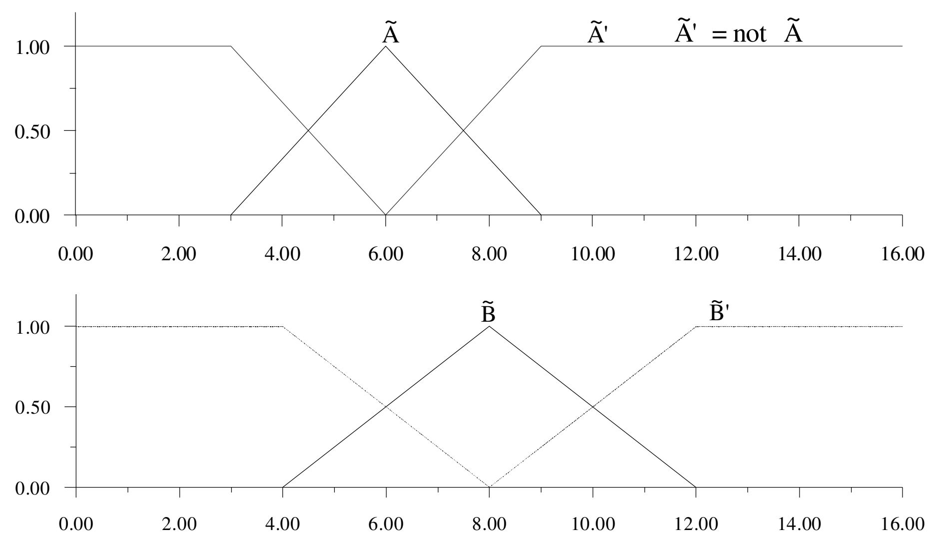
(a)


$\tilde{A}\ \tilde{A}'$
$\tilde{A}' =$ very $\tilde{A}$
1.00
0.50
0.00
0.00 2.00 4.00 6.00 8.00 10.00 12.00 14.00 16.00
$\tilde{B}\ \tilde{B}'$
1.00
0.50
0.00
0.00 2.00 4.00 6.00 8.00 10.00 12.00 14.00 16.00

(b)



(c)



(d)

Fig. 11 – Example to illustrate satisfaction of the four criteria (solid lines denote $\tilde{A}$ and $\tilde{B}$; dotted lines denote $\tilde{A}'$ and $\tilde{B}'$): (a) Criterion 1; (b) Criterion 2; (c) Criterion 3; (d) Criterion 4(II).

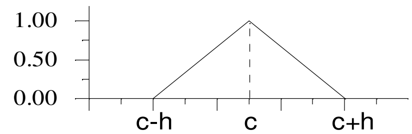


Fig. 12 – A membership function of isosceles-triangular shape.

(2) proof of Criterion 2:

Assume that the input fuzzy subset $\tilde{A}' =$ very $\tilde{A}$. We define the fuzzy subset "very $\tilde{A}$" as the one whose support is contained in that of $\tilde{A}$ and which has the same shape type as $\tilde{A}$, as shown in Fig. 11(b). Using Equations (5)-(8), we have

$$\begin{cases} t_{\tilde{B}} = t_{\tilde{A}} \times \dfrac{t_{\tilde{B}}}{t_{\tilde{A}}} = \dfrac{h_{\tilde{A}} \times h_{\tilde{B}}}{2 \times h_{\tilde{A}}} \\ u_{\tilde{B}} = u_{\tilde{B}} + \dfrac{t_{\tilde{B}}}{t_{\tilde{A}}}(u_{\tilde{A}} - u_{\tilde{A}}) = 4 \times c_{\tilde{B}} \\ v_{\tilde{B}} = v_{\tilde{A}} \times \dfrac{v_{\tilde{B}}}{v_{\tilde{A}}} = \dfrac{h_{\tilde{A}} \times h_{\tilde{B}}}{h_{\tilde{A}}} \\ w_{\tilde{B}} = w_{\tilde{A}} \times \dfrac{w_{\tilde{B}}}{w_{\tilde{A}}} = \dfrac{h_{\tilde{A}} \times h_{\tilde{B}}}{h_{\tilde{A}}} \end{cases} \Rightarrow \begin{cases} c_{\tilde{B}} = c_{\tilde{B}} \\ h_{\tilde{B}} < h_{\tilde{B}} \;\; (\dfrac{h_{\tilde{A}}}{h_{\tilde{A}}} < 1 \text{ and } \dfrac{h_{\tilde{B}}}{h_{\tilde{B}}} = \dfrac{h_{\tilde{A}}}{h_{\tilde{A}}}) \end{cases}$$

Hence the output fuzzy subset $\tilde{B}' =$ very $\tilde{B}$.

(3) proof of Criterion 3:

Assume that the input fuzzy subset $\tilde{A}' =$ more or less $\tilde{A}$. We define the fuzzy subset "more or less $\tilde{A}$" as the one whose support contains that of $\tilde{A}$ and which has the same shape type as $\tilde{A}$, as shown in Fig. 11 (c). Using Equations (5)-(8), we have

$$\begin{cases} t_{\tilde{B}} = t_{\tilde{A}} \times \dfrac{t_{\tilde{B}}}{t_{\tilde{A}}} = \dfrac{h_{\tilde{A}} \times h_{\tilde{B}}}{2 \times h_{\tilde{A}}} \\ u_{\tilde{B}} = u_{\tilde{B}} + \dfrac{t_{\tilde{B}}}{t_{\tilde{A}}}(u_{\tilde{A}} - u_{\tilde{A}}) = 4 \times c_{\tilde{B}} \\ v_{\tilde{B}} = v_{\tilde{A}} \times \dfrac{v_{\tilde{B}}}{v_{\tilde{A}}} = \dfrac{h_{\tilde{A}} \times h_{\tilde{B}}}{h_{\tilde{A}}} \\ w_{\tilde{B}} = w_{\tilde{A}} \times \dfrac{w_{\tilde{B}}}{w_{\tilde{A}}} = \dfrac{h_{\tilde{A}} \times h_{\tilde{B}}}{h_{\tilde{A}}} \end{cases} \Rightarrow \begin{cases} c_{\tilde{B}} = c_{\tilde{B}} \\ h_{\tilde{B}} > h_{\tilde{B}} \;\; (\dfrac{h_{\tilde{A}}}{h_{\tilde{A}}} > 1 \; and \; \dfrac{h_{\tilde{B}}}{h_{\tilde{B}}} = \dfrac{h_{\tilde{A}}}{h_{\tilde{A}}}) \end{cases}$$

Therefore, the output fuzzy subset $\tilde{B}' =$ more or less $\tilde{B}$.

(4) proof of Criterion 4(II):

When the input fuzzy subset $\tilde{A}' =$ not $\tilde{A}$, we can obtain $\tilde{B}' =$ not $\tilde{B}$. In the proof of this criterion, the parameters of fuzzy subset "not $\tilde{A}$" to be solved are the same as those of the fuzzy subset $\tilde{A}$. Hence the proof is similar to that for Criterion 1.

2.4. Fuzzifier and Defuzzifier

In rule matching, if the input is a crisp value, we fuzzify it to a fuzzy subset, $\tilde{A}'$, based on the membership-function shape of the antecedent $\tilde{A}$ of the fired rule. $\tilde{A}'$ is assumed to be of the same shape as $\tilde{A}$, and the grade of membership at x=the crisp input value is 1. If the antecedent $\tilde{A}$ is of trapezoid shape, then in $\tilde{A}'$ the grade of membership from $x=c$ to d is 1. If $\tilde{A}$ is of triangular or Π shape, then in $\tilde{A}'$ the grade of membership at $x=c$ is 1. See Fig. 13 for examples.

In many applications, we frequently define linear membership functions as those shown in Fig. 14(a). They are not of trapezoid shape. In the proposed method, we assume that the fuzzy subsets should be of trapezoid, triangular, or Π shape. Hence, in this situation we add a right side or a left side to the membership function such that it becomes a trapezoid, as shown in Fig. 14(b). The value of r or l is respectively set to be a very large or very small one in the x-axis.

After fuzzy inference, we will have a set of output fuzzy subsets $\tilde{B}'_j$, as shown in Fig. 15. To produce a crisp output value, we first use the center of gravity defuzzification method [20-23] to transform $\tilde{B}'_j$ to crisp values b_j for j=1, 2, …, m.

Next, we weight them with the scalar weights w_j for j=1, 2, …, m and sum them to produce the crisp output value b:

$$b = \sum_{j=1}^{m} w_j b_j$$

The weight w_j is determined according to the distance d_j between the crisp input value x' and the antecedent fuzzy subset of the jth rule. If the antecedent is of trapezoid shape, as shown in Fig. 4(a), the value of d_j is set to abs(x'-((c+d)/2)). If the antecedent is of triangular or Π shape, as shown in Figs. 4(b) and 4(c), the value of d_j is set to abs(x'-c). If x' is within the support of the antecedent of the jth fuzzy rule, we set the weight w_j to 1. Otherwise, we set w_j to $1/(d_j^3)$. When the given input is originally a fuzzy subset $\tilde{A}'$, the computation of d_j is as shown in Fig. 13.

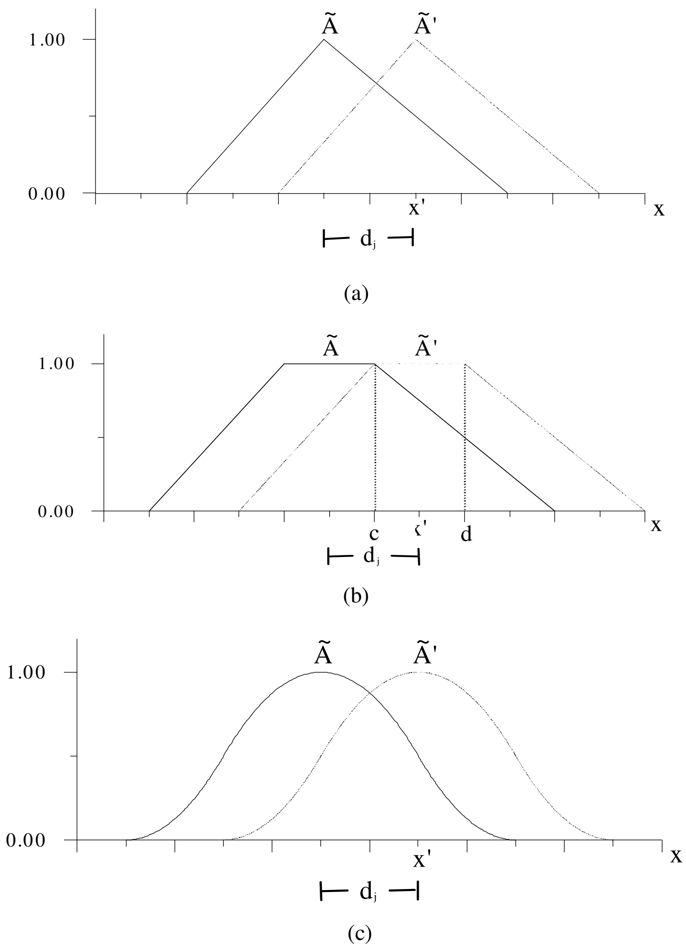
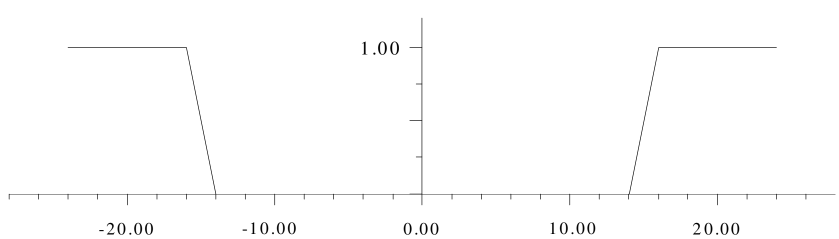
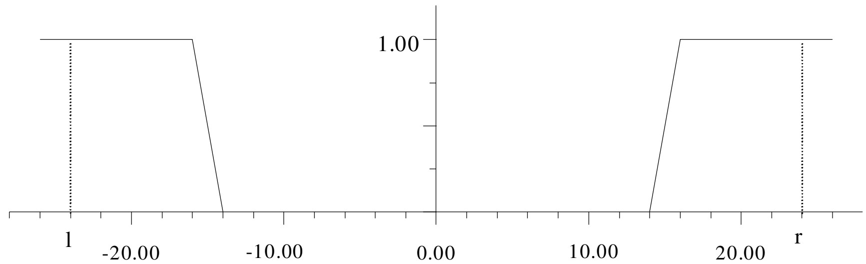


Fig. 13 – Fuzzifying a crisp input value *x'* based on the shape of $\tilde{A}$: (a) $\tilde{A}$ is of triangular shape; (b) $\tilde{A}$ is of trapezoid shape; (c) $\tilde{A}$ is of Π shape.

That is, $\tilde{A}'$ is regarded as a crisp value (=*x'*). And thus the computation of d_j is as mentioned above.



(a)



(b)

Fig. 14 – Linear membership functions.

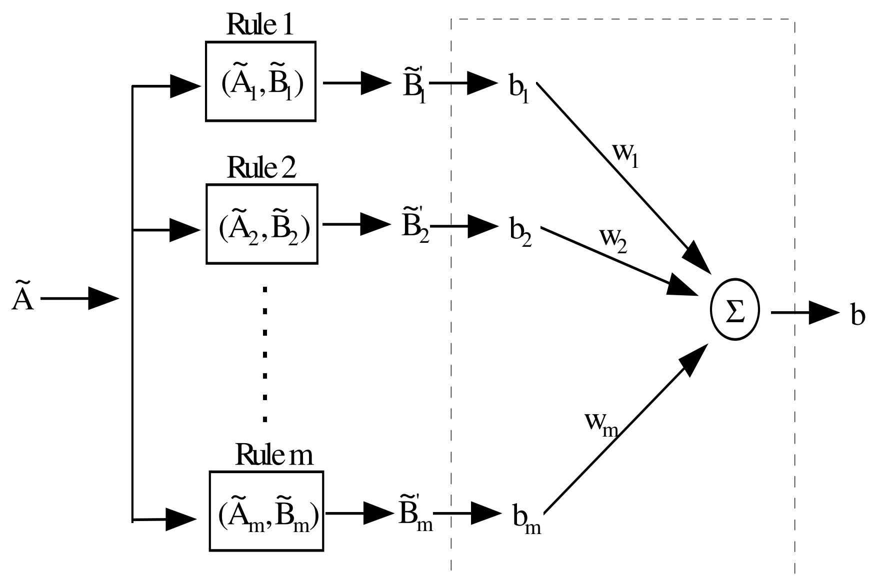


Fig. 15 – Defuzzification process.

3 Experimental Results

In this section, experimental results are presented to show the feasibility of the proposed approach. We use two examples to compare the results obtained by the proposed inference method and the Mamdani's method. One is a simple single-input single-output example, and the other is a truck backer-parking controller.

3.1. A Simple Example

In this section, a simple single-input single-output 3-fuzzy-rules example is used to compare the proposed inference method with the Mamdani's method. The fuzzy rules used in the example include:

If *x* is negative, then *y* is small.
If *x* is zero, then *y* is medium.
If *x* is positive, then *y* is large.

Fig. 16 shows membership-function graphs of the fuzzy subsets in the rules. The curves in Fig. 17 show the relationship between the inference output *y* and the input *x* obtained by the proposed and the Mamdani's methods. From the figure, we find that the same inference output is obtained by the Mamdani's method for input *x* between -10 and -5. It seems not reasonable. The output obtained by the proposed method is more reasonable, and better matches the human intuition.

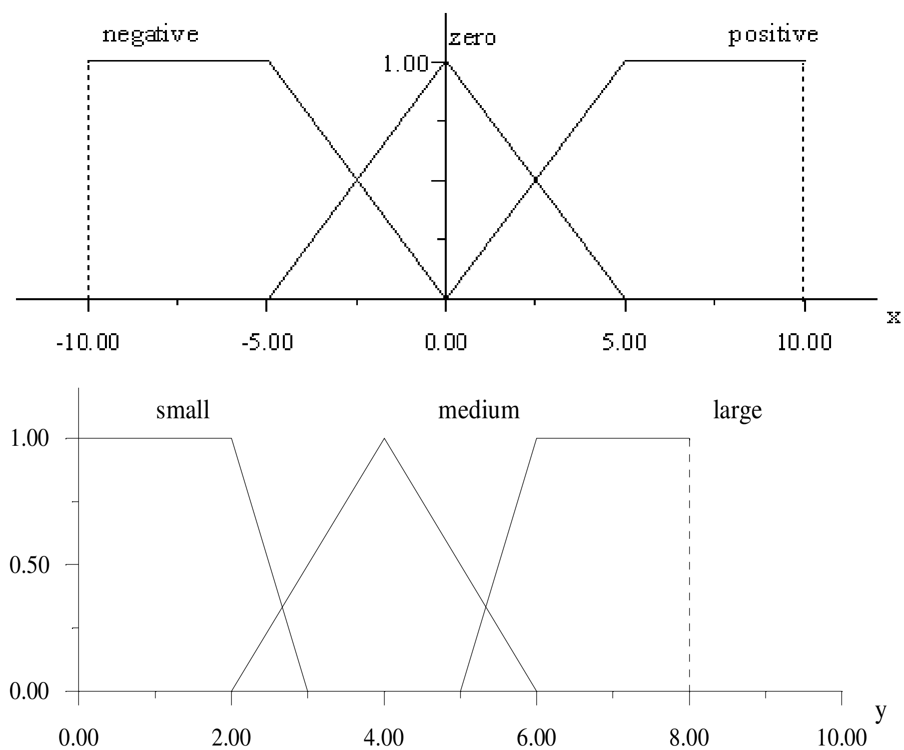


Fig. 16 – Fuzzy membership functions used in the single-input single-output example.

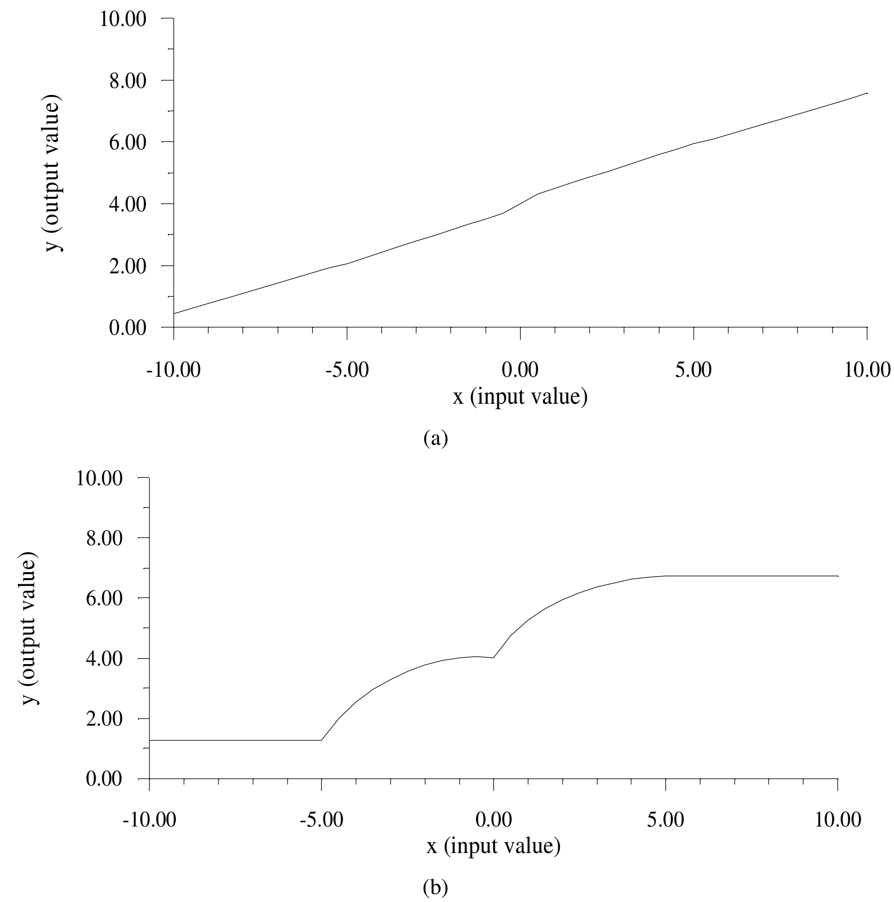


Fig. 17 – Inference results obtained: (a) by the proposed method; (b) by the Mamdani's method.

3.2. A Truck Backer-Parking Controller

In this section, we simulate a truck backer-parking controller to compare our proposed inference method with the Mamdani's method. The diagram of the truck and variables used are shown in Fig. 18. Our goal is to back up a truck to arrive at a loading dock at a right angle and to align the position (x, y) of the truck to the desired loading dock (50, 100). The variable ϕ denotes the angle of the truck with respect to the horizontal. The position of the rear center of the truck is (x, y). The variable θ denotes the steering angle of the truck. The input variables are ϕ and x, and the output variable is θ, which is also the control variable. The fuzzy-set values of the fuzzy variables are shown in Table 2.

The membership functions and fuzzy rules are defined as shown in Fig. 19 and Table 3, respectively [24].

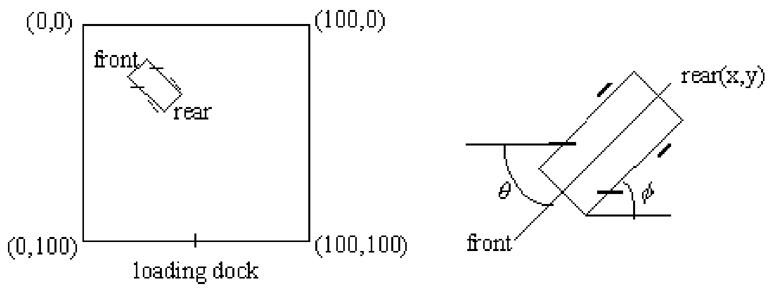
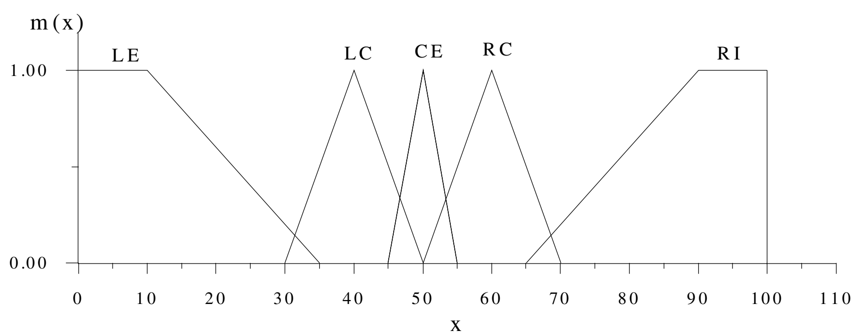
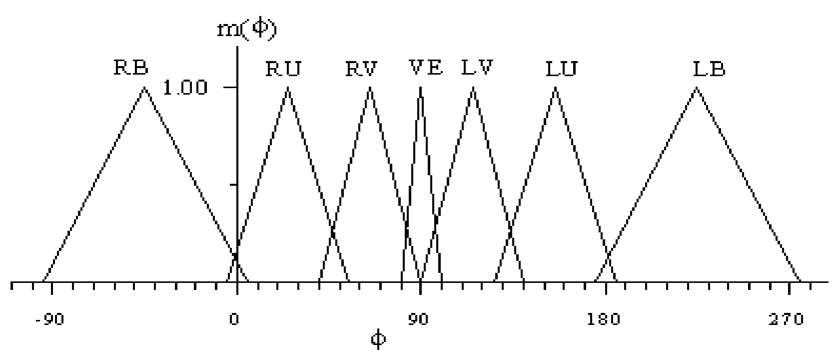


Fig. 18 – Diagram of the truck and loading zone.

Table 2 The fuzzy-set values of the fuzzy variables ϕ, x, and θ.

Angle ϕ	x-position x	Steering-angle θ
RB: Right Below	LE: Left	NB: Negative Big
RU: Right Upper	LC: Left Center	NM: Negative Medium
RV: Right Vertical	CE: Center	NS: Negative Small
VE: Vertical	RC: Right Center	ZE: Zero
LV: Left Vertical	RI: Right	PS: Positive Small
LU: Left Upper		PM: Positive Medium
LB: Left Below		PB: Positive Big





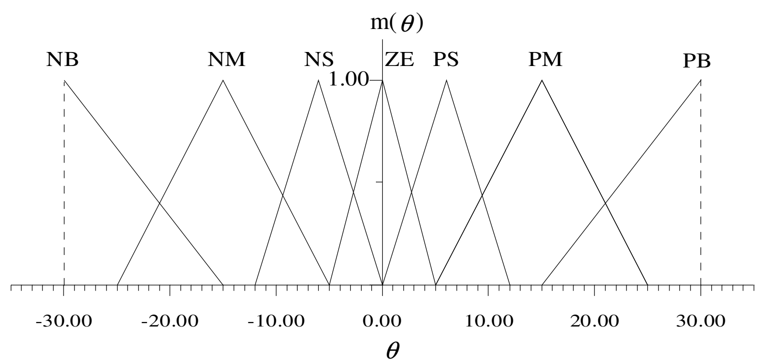


Fig. 19 – Fuzzy membership functions for fuzzy subsets used in the truck backer-parking controller.

Table 3 Fuzzy rules for the truck backer-parking controller.

ϕ \ x	LE	LC	CE	RC	RI
RB	1 PS	2 PM	3 PM	4 PB	5 PB
RU	6 NS	7 PS	8 PM	9 PB	10 PB
RV	11 NM	12 NS	13 PS	14 PM	15 PB
VE	16 NM	17 NM	18 ZE	19 PM	20 PM
LV	21 NB	22 NM	23 NS	24 PS	25 PM
LU	26 NB	27 NB	28 NM	29 NS	30 PS
LB	31 NB	32 NB	33 NM	34 NM	35 NS

Figs. 20 and 21 show sample truck trajectories obtained by the proposed inference method and the Mamdani's method, respectively. In the figures, the initial positions (x, y, ϕ) of the truck are (a) (80, 20, 120), (b) (50, 20, 90), (c) (30, 10, 0), (d) (30, 10, 220), (e) (30, 40, -80), and (f) (60, 20, 70). In these experiments, the truck trajectories produced by the proposed inference method are similar to those produced by the Mamdani's method. The surfaces in Fig. 22 show the relationship of the inference output θ and the inputs ϕ and x, where the initial position of the truck is (30, 20, 50). The surface in Fig. 22 (b) has steeper jumps than that in Fig. 22 (a). The control surface obtained by the proposed method is smoother than that obtained by the Mamdani's method. We study robustness of our proposed inference method and the Mamdani's method by removing randomly fuzzy rules from the system. In the Mamdani's inference method, removing rules may result in the case that no rule is fired. In this situation, no inference output can be obtained. To continue the control process, we assume that in this case the truck follows the last action inferred in the previous cycle. However, this situation will not happen in the proposed method. This is because we always have an inference output for each fuzzy rule.

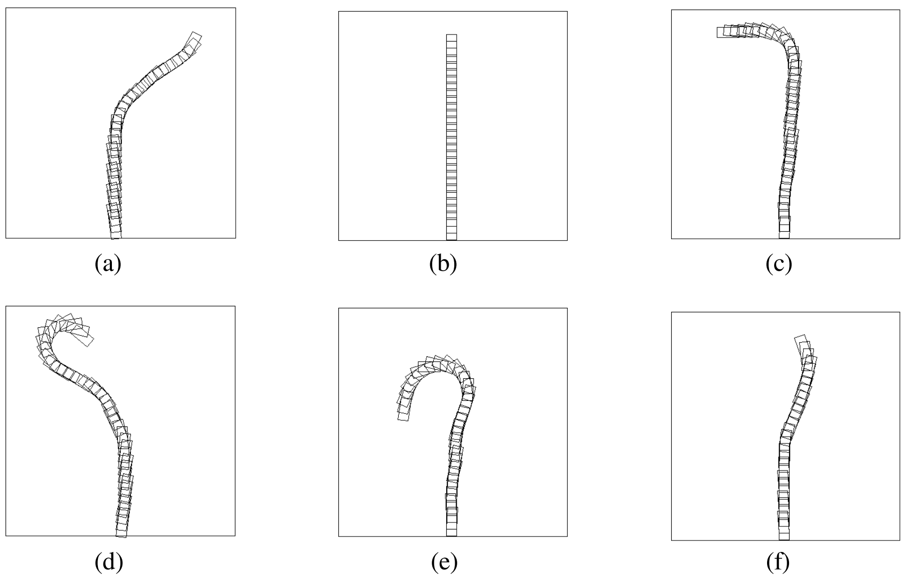


Fig. 20 Sample truck trajectories obtained by the proposed inference method.

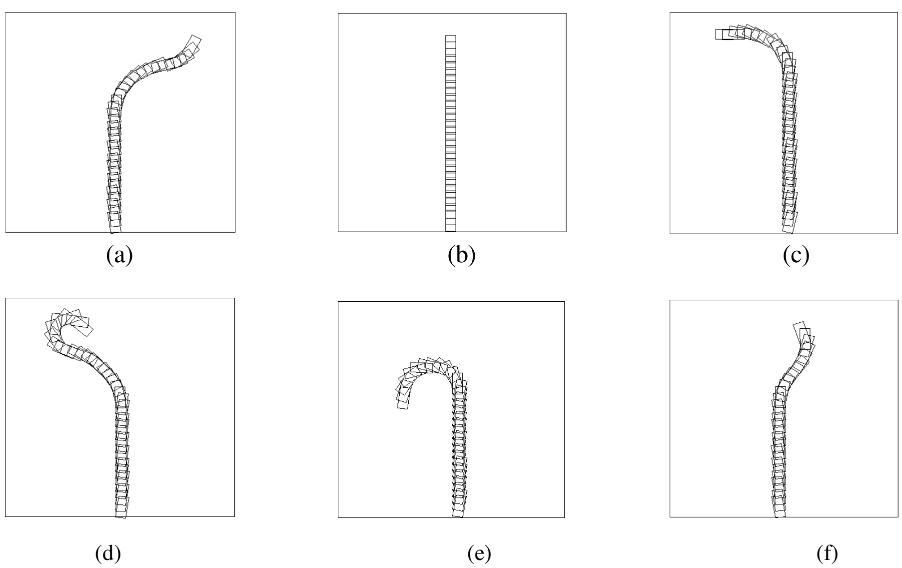
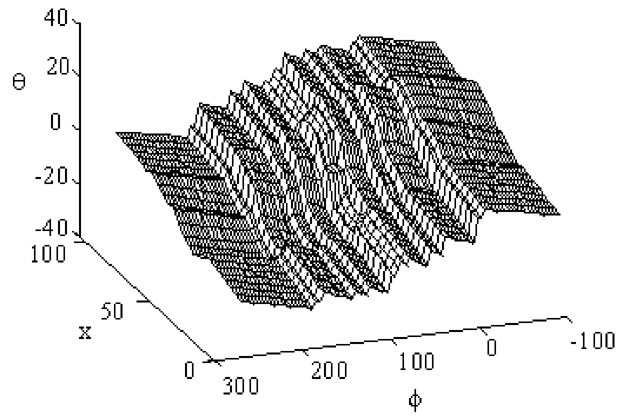
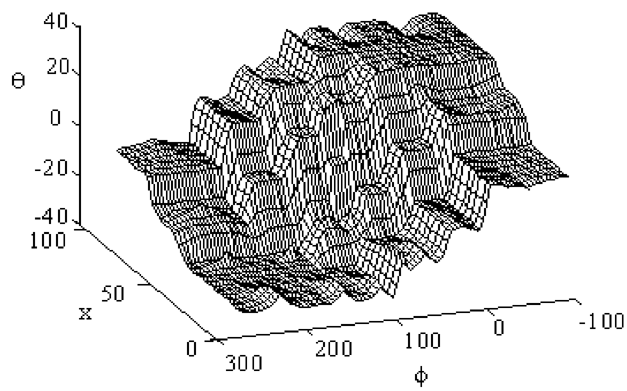


Fig. 21 Sample truck trajectories obtained by the Mamdani's method.



(a)



(b)

Fig. 22 Control surfaces obtained: (a) by the proposed inference method; (b) by the Mamdani's method.

The experimental results obtained by our proposed method are shown in Fig. 23 and those obtained by the Mamdani's method are shown in Fig. 24. In these figures, the initial position of the truck is (30, 20, 50). In Figs. 23 (a) and 24 (a), no fuzzy rule is removed. In Figs. 23 (b) and 24 (b), two fuzzy rules (23, 30) are removed. In Figs. 23 (c) and 24 (c), four fuzzy rules (4, 25, 26, 34) are removed. In Figs. 23 (d) and 24 (d), six fuzzy rules (1, 10, 26, 29, 30, 32) are removed. In Figs. 23 (e) and 24 (e), eight fuzzy rules (4, 5, 7, 11, 17, 18, 24, 33) are removed. In Figs. 23 (f) and 24 (f), ten fuzzy rules (1, 9, 13, 16, 17, 19, 20, 24, 27, 35) are removed. In Figs. 23(g) and 24 (g), twelve fuzzy rules (1, 2, 5, 6, 13, 15, 16, 17, 18, 19, 33, 34) are removed. In Figs. 23 (h) and 24 (h), fourteen fuzzy rules (1, 2, 4, 10, 15, 16, 17, 20, 21, 23, 24, 25, 28, 32) are removed. In Figs. 23 (i) and 24 (i), sixteen fuzzy rules (3, 4, 6, 9, 15, 16, 17, 18, 19, 20, 22, 23, 24, 27, 29, 35) are removed. In Figs.

23 (j) and 24 (j), eighteen fuzzy rules (1, 3, 5, 6, 7, 9, 10, 14, 16, 19, 20, 22, 23, 27, 29, 30, 34, 35) are removed. In Figs. 23 (k) and 24 (k), twenty fuzzy rules (2, 4, 5, 6, 7, 8, 9, 11, 13, 16, 18, 19, 20, 21, 22, 23, 28, 30, 31, 34) are removed. In Figs. 23 (l) and 24 (l), twenty two fuzzy rules (1, 4, 5, 7, 8, 10, 12, 14, 15, 16, 17, 18, 19, 21, 23, 24, 27, 28, 31, 33, 34, 35) are removed.

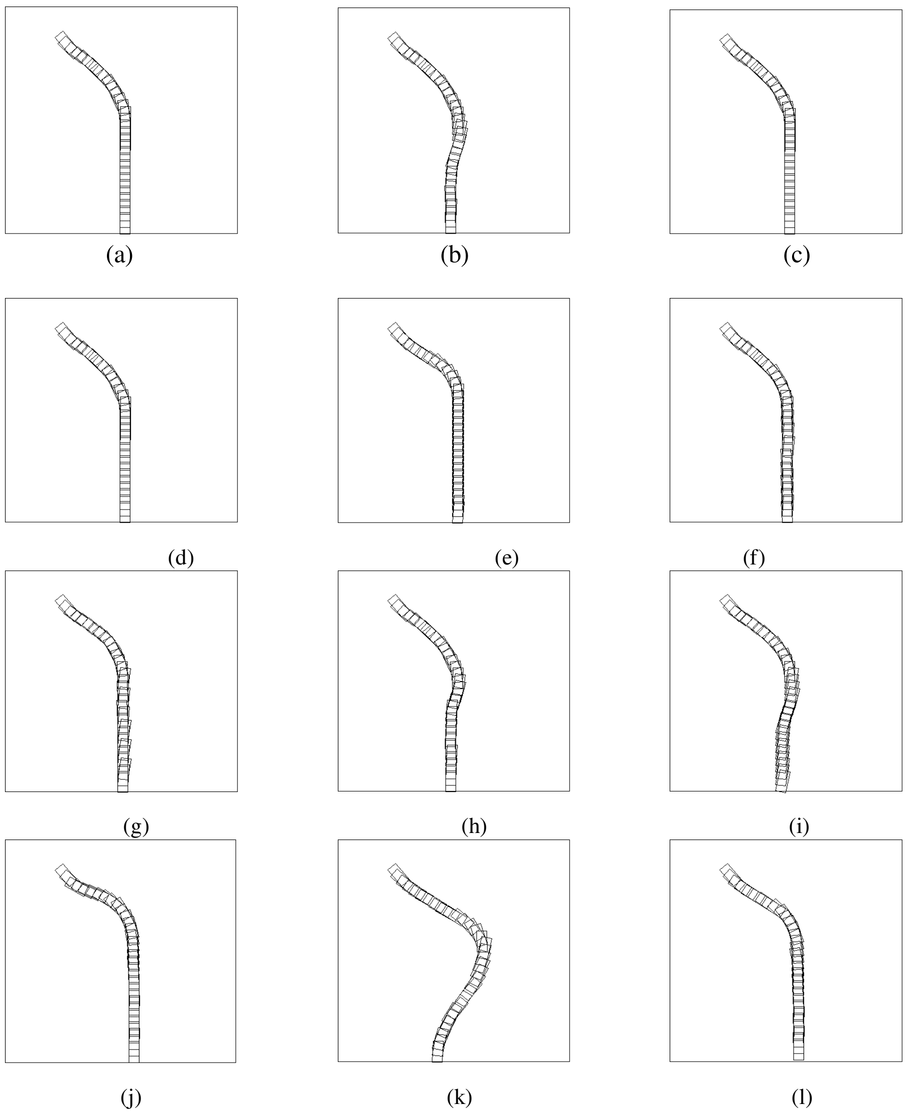


Fig. – 23 Sample truck trajectories obtained by the proposed method when rules are randomly removed.

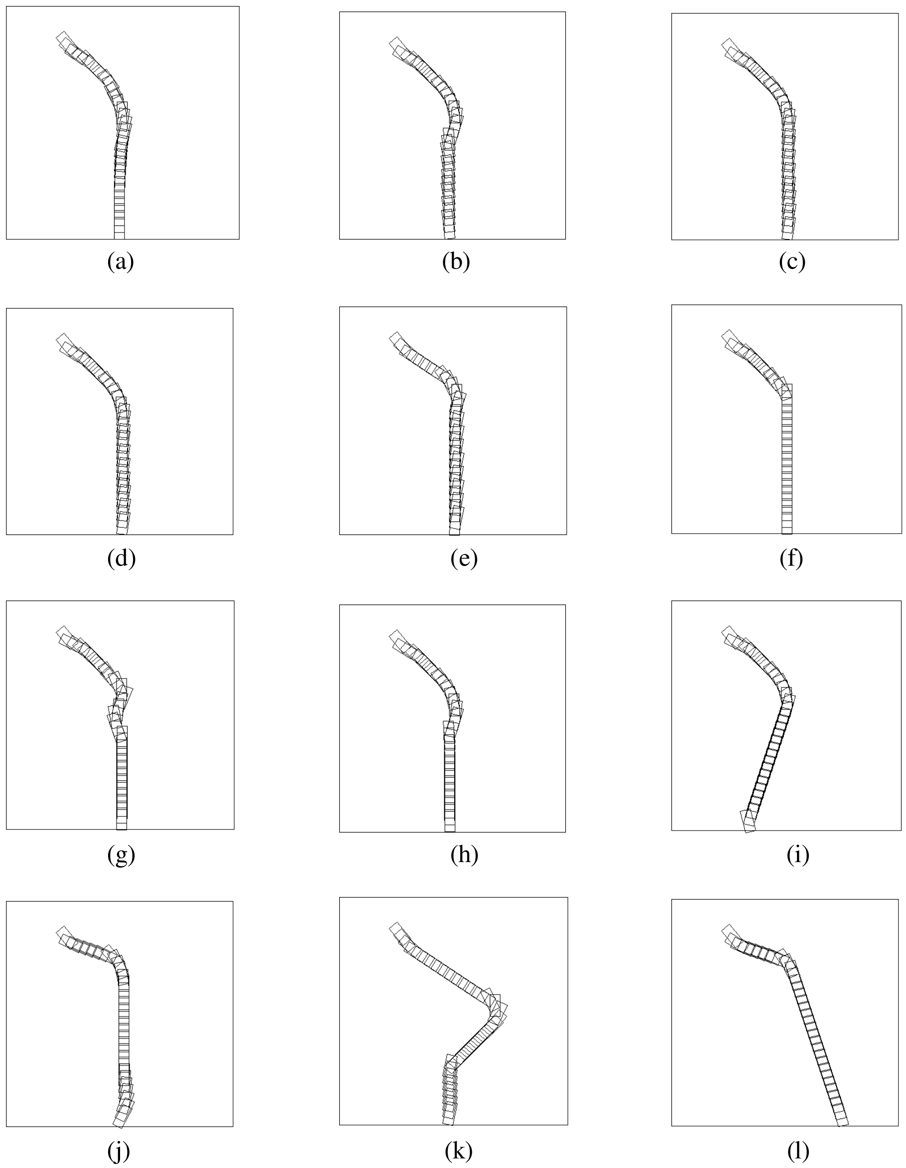
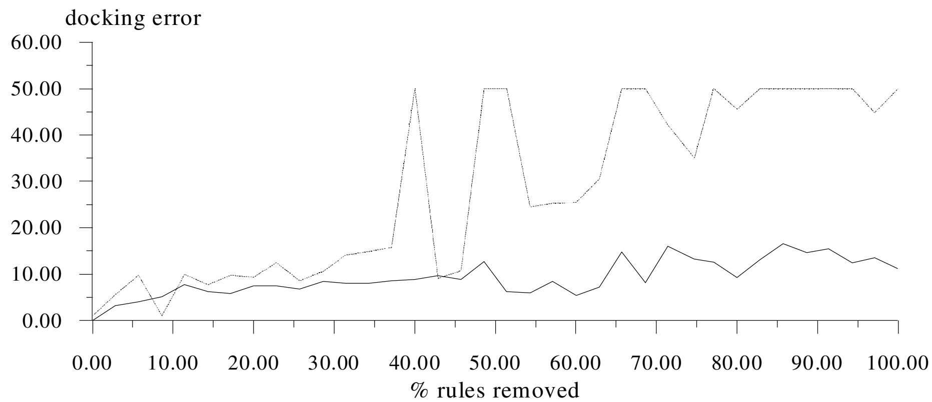
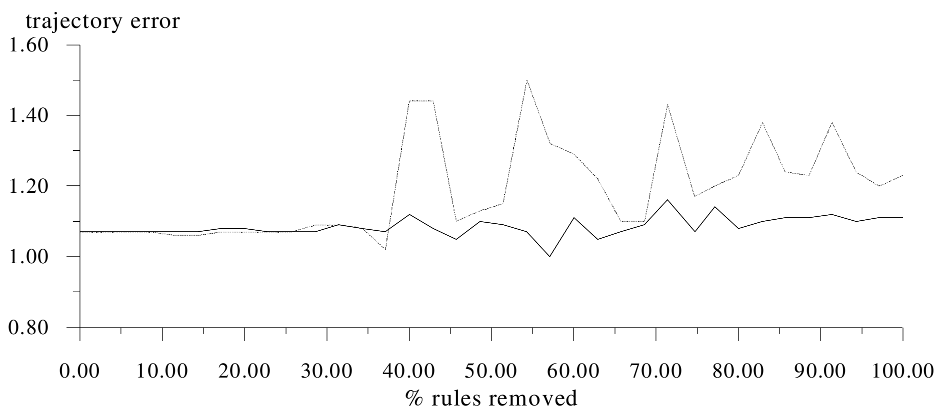


Fig. 24 – Sample truck trajectories obtained by the Mamdani's method when rules are ranmomly removed.

All the rules removed are selected randomly. The experimental results show that our proposed inference method performs better than the Mamdani’s method. In the Mamdani’s method, if a large percentage of fuzzy rules are removed, the truck trajectory may not be smooth. The graphs of the docking error and trajectory error corresponding to the percentage of rules removed for the proposed method and the Mamdani's method are shown in Fig. 25.



(a)



(b)

Fig. 25 Comparison of robustness (the solid line for the proposed method and the dotted line for the Mamdani's method): (a) the docking error; (b) the trajectory error.

The docking error is the Euclidean distance from the actual final position (x, y, ϕ) to the desired final position (x_f, y_f, ϕ_f), and the trajectory error is the ratio of the actual trajectory length of the truck to the straight-line distance from the initial position to the loading dock. From the figure, it is clear that our proposed inference method is better.

4 Conclusions

In this study, we have proposed a new fuzzy inference method, which closely imitates the intuition of human reasoning. The output fuzzy subset is derived based on the spatial relationship of related fuzzy subsets. The membership function of the output fuzzy subset has the same shape type as that of the input fuzzy subset. Moreover, the location of the output fuzzy subset matches the human intuition. The advantages of our proposed inference method compared to the Mamdani's method are that smoother input-output curve (or surface) and greater robustness are obtained in our method. Experiments have been performed to show these.

Further research may be directed to extending the proposed inference method to produce other shapes of output fuzzy subsets, for example, by the resolution principle [25]. In this study, only two examples are used to show the performance of our method. In the future, the proposed inference method will be applied to more complex applications such as intelligent vehicle navigation and collision avoidance.

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