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The Behavior of Continued Fractions-based Networks and T – Ganglions

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Abstract: The behavior of continued fractions neural networks and t-ganglia network is analyzed in the present paper. The results of the analysis on the dynamic of continued fraction-based system/network are presented. The goal was to use the analysis made on continued fractions-based neurons in different cases of neural network topologies to find out what type of dynamic these networks respond with. The analysis was made using nonlinear and statistic tools. The results obtained show a nonlinear dynamic in most of the cases, and quite intricate for the CMN topology. Possible applications are here mentioned.

1. Introduction

Continued fractions, as a mathematical model, have been proven to be a good model for neural networks design, modeling and analysis. This paper is the continuation of some preliminary analysis on the continued fractions class based networks and t-ganglions made in some previous papers [1-3][E1]. Using these fractions several neural network topologies have been further analyzed. The analysis results obtained have showed that neural networks with continued fractions-based neurons have an interesting, rich, nonlinear behavior in most of the cases. Several tools [4] for nonlinear dynamic analysis have been used in the analysis. Among them we name a few like bifurcation diagram, the phase diagram, the calculation of the Lyapunov coefficient, the biparametric Lyapunov map, and many others.

The information introduced at the input propagates in time, changing the outputs of the network. In most cases regarding the various network topologies analyzed and regarding the fact that the input information values have been several times varied or changed a rich and nonlinear behavior was obtained for this type of fractions. The slightest variation of this parameter has propagated to the outputs of the network influencing their values in a significant way. A[E2] radial basis characteristic function has been used in for the neurons of similar networks for comparison.

The paper is structured as follows. In section 2, the network topologies are present. In section 3 a presentation of the fractions used for implementation has been made; in section 4 the results of the analysis made are showed and discussed. In the last section of the article, namely section 5, some conclusions and remarks are made, suggesting possible applications for this type of fractions.

2. Neural Network Topology

Several network topologies like Linear Infinite1 Lattice and Semi-planary Infinite[2]2Coupled Map Network (belonging to the Coupled Map Networks/Lattice Class), and Ganglia like network [5-7] were used in simulation.

On the first ones a preliminary analysis has been made in a previous paper, the results obtained being encouraging. This type of network topology has several proven properties [8-10] regarding the occurrence of the self-organization process present at the network outputs. The CMN/L have been proposed initially by Kaneko ODE [11] (ODE model – Ordinary Differential Equations – discretized) and used later by physicians, due to the properties of the network. They have been using them to model various physic processes like liquid flow in a tube, or propagation of air turbulence, genetics, were the organization of information occur.

Self-organization, is a phenomenon that is present in Coupled Map Networks/Lattice, and implies process information aggregation, using knowledge and data, to produce new knowledge to describe the modeled process. This feature can by extended in modeling of any process that we suspect that it has a non-trivial dynamic, namely a nonlinear one. These properties have been demonstrated in many recent papers due to the interest of modeling social and biological processes/systems [8,9][15].

The neural networks ganglia like, and the extensions (generalizations) introduced in [9,10][12], mentioned here and have strong nonlinear behavior, being able to serve as a realistic and easy to implement models, software and hardware.

The properties of these networks – namely generating and/or classifying classes of patterns, the occurrence of the self-organization phenomenon on the output layer – can be used to model some processes involving information propagation from a source that feeds the system permanently with information. Among them we mention the physical, demographically, biological, social, economical processes, etc. Another area of applications – non-linear measurements – use such systems due to their properties, following the principle presented in [2,3].

2.1. Coupled Map Networks/Lattice

In figure 1-a) is represented an example of a linear infinite lattice – 1D. A single line, layer or row can represent the network, with infinite number of neurons. The propagation is from one neuron to the next neuron, the influence of the "behind" neurons being present.

The equation (1) describes the network from figure 1–a):

$$x_{n+1}^{i+1} = f\left(x_n^{i-1}, x_n^i\right) \quad \Leftrightarrow \quad x[i+1][n+1] = f\left(x[i-1][n], x[i][n]\right) \tag{1}$$

1 number of neurons is infinite

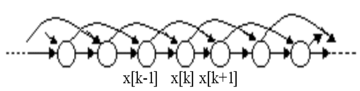
2 number of neurons from one layer is infinite

where “n” represents the number of the neuron in a linear lattice.

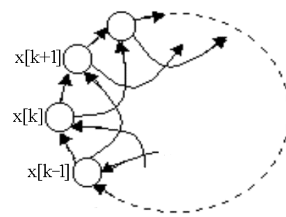
Function f is the characteristic function. To compute the value of the output for the neuron “k”, one can write:

$$x_{n+1}^{i+1} = f\left(x_n^{i-1}, x_n^{i}\right) = a \cdot x_n^{i-1} + b \cdot x_n^{i} \tag{2}$$

In figure 1-b) is represented a ring lattice, with uni-directional connection, and having a vicinity of order 2. The type of information propagation is practically the same, only with the remark that the topology becomes ring topology, and the first neuron of the network becomes the last in the same time. The equation remains identical. The input signal is introduced as initial condition of the system, the information constantly feeding the system. This information is actually the excitation of the input neuron. The propagation process can be spatial domain propagation, or temporal domain propagation, depending on the application and/or the process that we propose to model.

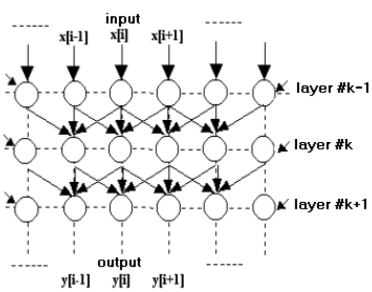


a) linear infinite lattice

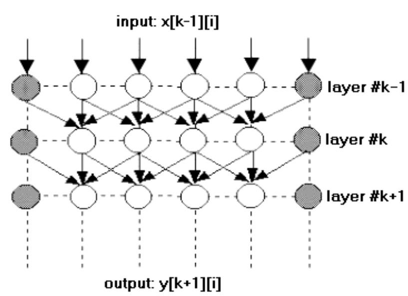


b) ring lattice, with uni-directional connection

Fig. 1 – Examples of lattice – 1D (after [5][6])



a) Infinite Semi-planary CMN

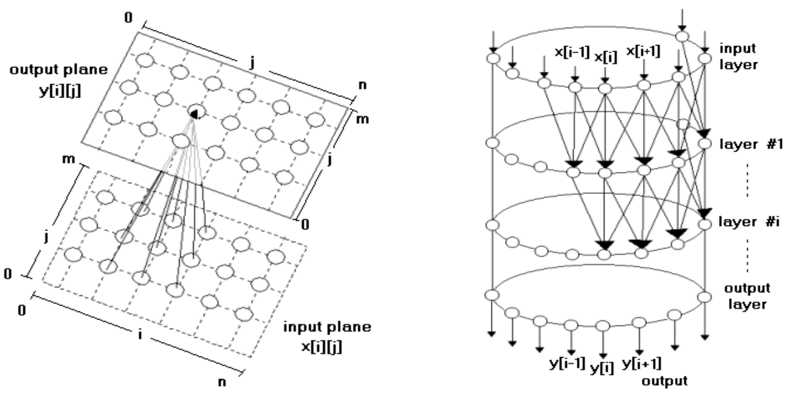


b) Boundary conditioned Semi-planary CMN

Fig. 2 – Semi-planary CMN (after [8][9])

In figure 2, are depicted two semi-planary CMN/L, one of them having boundary conditions, meaning that the neurons from margins an have fixed values, or they simply keep the same values for those margin neurons from the previous row neuron on the same column of the network. In figure 3 – a) a square bi-planary CMN/L is presented.

Another version of a planary CMN/L [8,9], is presented in figure 3-b) where we have “*n*” planes, disposed on parallel circles, on a cylinder. The information propagation for this type of network topology is from plane to plane (circle -> circle/layer “*k*”->layer “*k*+1”). The order of vicinity for the network is *N*=3. The notations are *x*[*i*] for input signal of the neuron “*i*” belonging to the input layer, and with *y*[*i*] the output signal of the neuron “*i*”. The layers between the input and the output layer are usually called hidden layers and they are holding the processing elements of the network (neurons for processing the information propagated from one layer to another).



a) bi-planary CMN **b)** n-planary CMN

Fig. 3 – Planary Coupled Map Networks/Lattice (after [8,9])

2.2. T-Ganglia Networks

In figure 2 is presented an example of a T-ganglia network, uni-directional. Here the previous value of the neuron of order “*k*” from the previous step, “*p-1*” appears in the equation for the neuron of order “*k*” from the step “*p*”, besides the value of the neuron “*k-1*”. In the right side we have the discretized equation:

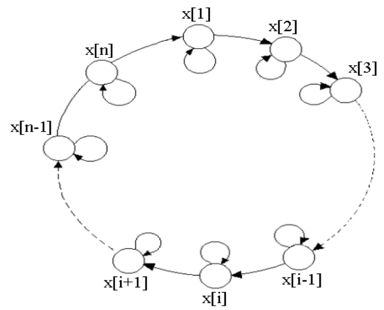
$$x_p^k = f\left(x_p^{k-1}, x_p^k\right) \quad \Leftrightarrow \quad x[k][p] = f\left(x[k][p-1], x[k-1][p]\right) \tag{3}$$

In figure 4-a) is presented an example of ganglia like network, uni-directional, with feedback loop. Here the previous value of the neuron of order “*k*” from the previous step, “*p-1*” appears in the equation for the neuron of order “*k*” from the step “*p*”, besides the value of the neuron “*k-1*”. Equation 3 becomes:

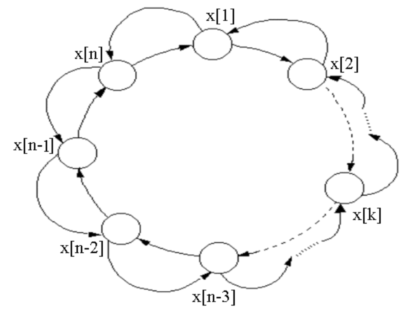
$$x_p^k = f\left(x_p^{k-1}, x_p^k\right) \tag{4}$$

For the discretized case, equation 4 becomes:

$$x[k][p] = f\left(x[k][p-1], x[k-1][p]\right) \tag{5}$$



a) T-Ganglia network, uni-directional, with feed-back



b) T-Ganglia network – 1D, bi-directional, vicinity order 1

Fig. 4 – Examples of generalizations for neural networks, ganglia like [6][8]

In figure 4-b) is represented a ganglia like network, bi-directional, with simple vicinity.

$$x_p^k = f\left(x_{p-1}^{k+1}, x_p^{k-1}\right) \tag{6}$$

or

$$x[k][p] = f\left(x[k+1][p-1], x[k-1][p]\right) \tag{7}$$

One can note how easy is to generalize such a network, obtaining a ganglia like network with an order for the vicinity of a neuron sufficiently large, necessary if we want to model a very complex process, with a non-trivial dynamic. The phenomenon of self-organization that (may) occur is very strong represented, the result being a very large class of (new) patterns.

3. Continued Fractions

Consider the quadratic equation:

$$x^2 - bx - 1 = 0 \tag{8}$$

Dividing by x we can rewrite it as:

$$x = b + \frac{1}{x} \tag{9}$$

Now substitute the expression for given by the right-hand side of this equation for in the denominator on the right-hand side:

$$x = b + \frac{1}{b + \frac{1}{x}} \tag{10}$$

We can continue this procedure indefinitely, to produce a never-ending staircase of fractions:

$$x = b + \cfrac{1}{b + \cfrac{1}{b + \cfrac{1}{b + \text{O}}}} \tag{11}$$

The general form of continued fraction of a number as:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \Lambda + \cfrac{1}{a_i + \cfrac{1}{\text{O}}}}}} \tag{12}$$

A general expression of the Continued Fractions used in the present analysis is written below. The point of departure is represented by the following recurrent function:

$$f_n = \cfrac{1}{1 + \cfrac{1}{f_{n-1}}} \text{ with } f_0 = \frac{1}{2}. \tag{13}$$

The generalization goes to the form:

$$f_n = f_n(\alpha, a, b, x_{n-1}, x_{n-2}, x_{n-3}), \tag{14}$$

For easy implementation the following generalization is made:

$$x_n = \alpha \cdot x_{n-1} + \frac{1}{1 - a \cdot x_{n-1}(1 - b \cdot x_{n-2})}. \tag{15}$$

Based on this definition the derivative was obtained in the following expressions, generalizing and making a replacement of the previous term step by step.

Then one can write replacing in the equation (12) the term x_{n-1}:

$$\begin{aligned} x_n &= \alpha \cdot x_{n-1} + \frac{1}{1 - a \cdot x_{n-1}(1 - b \cdot x_{n-2})} = \\ &\alpha \cdot \left[\alpha \cdot x_{n-2} + \frac{1}{1 - a \cdot x_{n-2} \cdot (1 - b \cdot x_{n-3})}\right] + \\ &+ \frac{1}{1 - a \cdot \left[\alpha \cdot x_{n-2} + \frac{1}{1 - a \cdot x_{n-2} \cdot (1 - b \cdot x_{n-3})}\right] \cdot (1 - b \cdot x_{n-2})} \end{aligned} \tag{16}$$

By replacing x_{n-2} in the equation above one can obtain the following expression:

$$x_n = \alpha\cdot\left\{\alpha\cdot\left[\alpha\cdot x_{n-3}+\frac{1}{1-a\cdot x_{n-3}\cdot\left(1-b\cdot x_{n-4}\right)}\right]+\frac{1}{1-a\cdot\left[\alpha\cdot x_{n-3}+\frac{1}{1-a\cdot x_{n-3}\cdot\left(1-b\cdot x_{n-4}\right)}\right]\cdot\left(1-b\cdot x_{n-3}\right)}\right\}+$$

$$+\frac{1}{1-a\cdot\left\{\alpha\cdot\left[\alpha\cdot x_{n-3}+\frac{1}{1-a\cdot x_{n-3}\cdot\left(1-b\cdot x_{n-4}\right)}\right]+\frac{1}{1-a\cdot\left[\alpha\cdot x_{n-3}+\frac{1}{1-a\cdot x_{n-3}\cdot\left(1-b\cdot x_{n-4}\right)}\right]\cdot\left(1-b\cdot x_{n-3}\right)}\right\}\cdot\left(1-b\cdot\left[\alpha\cdot x_{n-3}+\frac{1}{1-a\cdot x_{n-3}\cdot\left(1-b\cdot x_{n-4}\right)}\right]\right)}=\mathrm{K}$$

In the equation above the term x_{n-2} was replaced by the expression $\alpha\cdot x_{n-3}+\frac{1}{1-a\cdot x_{n-3}\cdot\left(1-b\cdot x_{n-4}\right)}$. The next step is to replace in the equation obtained the term x_{n-3} a.s.o. The recurrence present in the fraction is easy to follow, and the implementation was likewise.

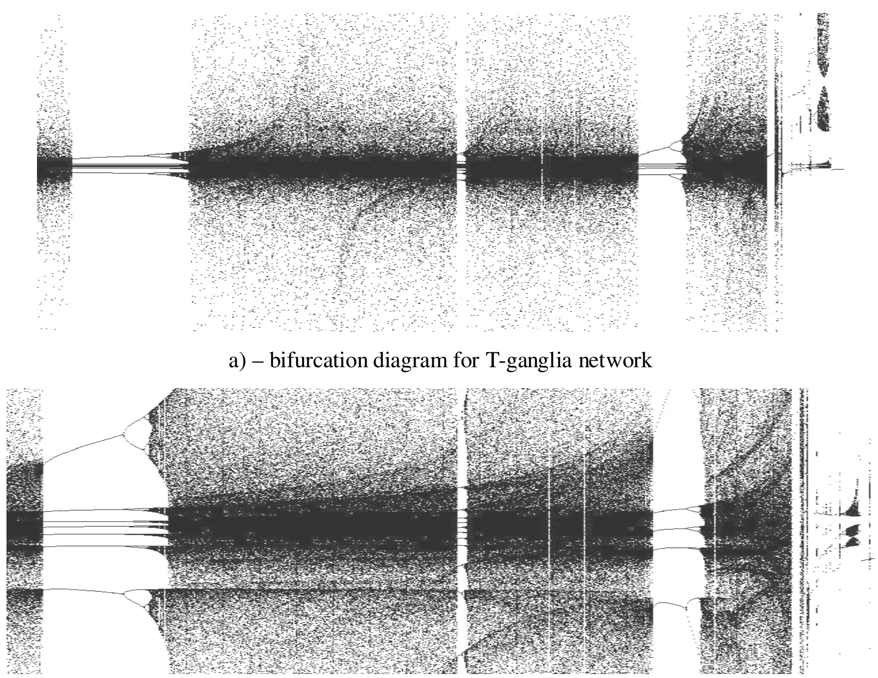
4. Simulation results

The bifurcation diagram showed in figure 5-a) is obtained for a linear lattice (figure 1-a) using an infinite ratio fraction as a characteristic function. The bifurcation diagram depicted in figure 5-b) has been obtained for a T-Ganglia network, using the same values for the fraction parameters (α, a, b). The behavior can easily be differentiate between the two network topologies, even if the dynamic is similar (nonlinear). One can observe that the complexity of the network dynamic depends on the network topology. For the first case, the propagation of the information flows from one neuron to the next neuron. In the case of this topology the cells of the network are linked one by one, so the information flows from neuron "*k*" to the neuron "*k*+1" so the evolution in time can only be feed-forward). In the second case appears a influence of the neurons from behind, so the information of the current neuron is influenced by the information from the 2 precedent neurons.

The information propagates from the neurons "*k*-2" and "*k*-1" to the neuron "*k*".

So in the case of this topology the information propagates feed-forward and in the same time is spreading. Due to these results we could find appropriate an approach with this type of network topology in modeling several economical or social processes. Definitely physical or biological processes can be modeled with this type of network, due to their "data flow" feature and spreading of data or self-organization occurrence in the process.

A major fact that we have observed in the behavior of the CMN/L topology – using various types of characteristic functions – that the phenomenon of self-organization and "spreading" occurs. The fractions proposed have their one characteristic regarding this feature [2][13]. From the bifurcation diagram we observe that the system has the tendency to enter in a nonlinear regime, then the behavior is at some point limited and the fraction's behavior enters in stable and then in a periodical regime, only to return in chaos.

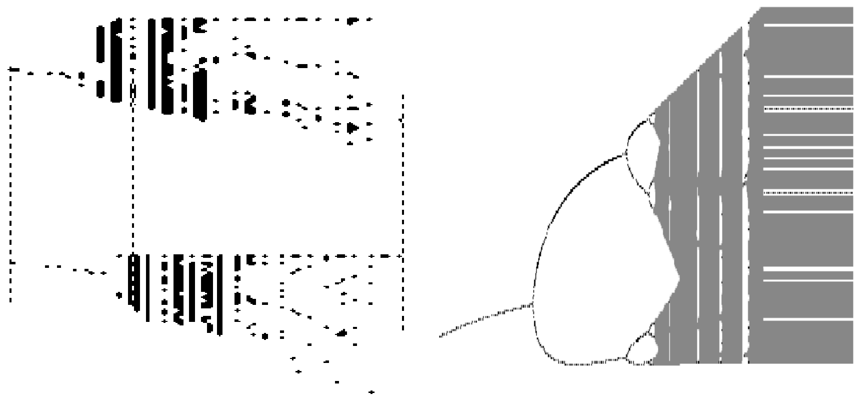


a) – bifurcation diagram for T-ganglia network

b) – bifurcation diagram for linear lattice (CMN/L)

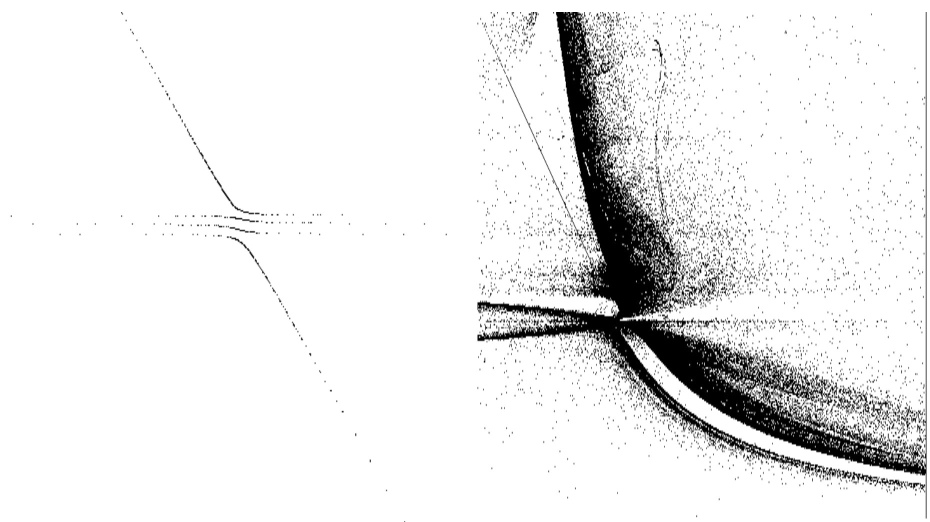
Fig. 5 – Bifurcation Diagram [8]

One can observe also the richness of the network behavior, for a CMN/L comparing to ganglia like network. For other types of characteristic functions the results is not that obvious (compare fig. 5 – bifurcation diagram for continued fractions based networks with fig. 6 – bifurcation diagram for RBF-based neurons bifurcation diagram) and Logistic Map bifurcation diagram.



RBF-based neurons [8] Logistic Map

Fig. 6 – Bifurcation Diagram

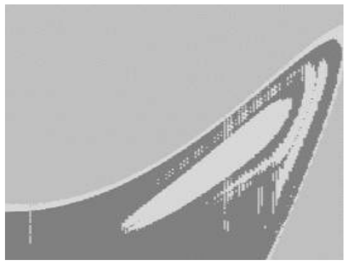


A – Phase diagram for continued fractions based T-ganglia network [13]

B – Phase diagram for continued fractions based CMN (order 3 recurrence)

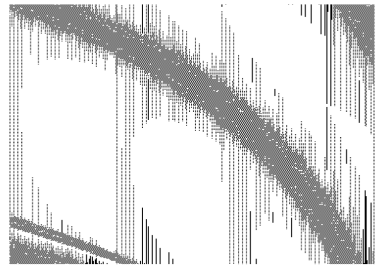
Fig. 7 – Phase diagrams

Lyapunov coefficients computed for this network with characteristic functions that involve feedback, and moreover are nonlinear equations, came out to be highly positive ($\lambda_i > 0.18, i > 1$). The correlation dimension computed was $D_C = 0{,}012658$, with an error $\varepsilon = 0{,}001325$ [11].



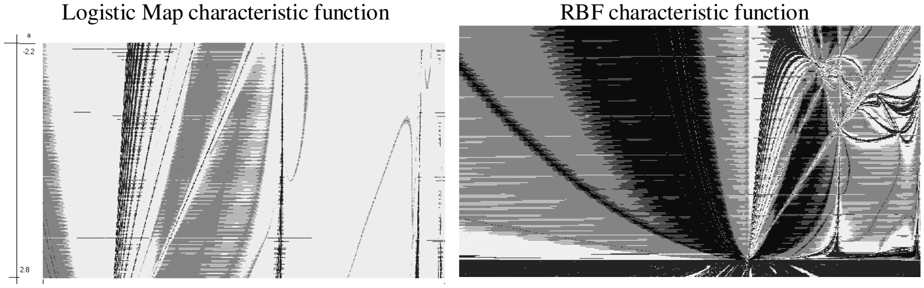
Bi-parametric Lyapunov for

Logistic Map characteristic function



Bi-parametric Lyapunov for

RBF characteristic function



Biparametric Lyapunov Map for continued fractions based

Coupled Map Networks /Lattice

Fig. 8 – Bi-parametric Lyapunov

After the analysis made using a CMN/L, one can see that the system has a pronounced chaotic dynamic. The network's dynamic behavior has proved to be very sensitive to the input parameters of the network, the slightest variation of the input signal, with a very small interval for the input values ($\pm 10^{\pm 3 -> 4}$).

The high sensitivity to the input signal parameters determines the properties of this kind of network (continued fractions based neurons in comparison to other known chaotic characteristic functions) selectivity, classification and pattern generation.

The figures above show in contrast the bi-parametric Lyapunov Map obtained for different characteristic functions but with the same network topology.

In the simulations made the parameters values were as follows:

- initial value of x was: $x = 0.2235$, then $x = 0.0468$
- parameter α was fixed to the value: $\alpha = 0.0058$, then to $\alpha = 0.0081$
- parameter β was fixed to the value: $\beta = 0.0031$, then $\beta = 0.0001$
- parameter a was varied in the interval: $a \in [-4.2;+2.8]$ at a step of 0.003, then 0.005
- parameter b was varied in the interval: $b \in [-4.2;+2.8]$ at a step of 0.003, then 0.005
- parameter $\varepsilon = 0.0001$

5. Conclusions

In the previous papers this type of fractions that present such interesting dynamic have been suggested – due to the fact that they are satisfying the necessities required for modeling social, physics and/or biological processes.

The topologies used in analysis are simple exactly to demonstrate the fact that the behavior of this type of fractions is non-trivial in the simplest conditions. This characteristic function is much richer in comparison to others, the use of the same topologies being made only to underline the features of this type of fractions. From the analysis of CMN/L for different characteristic functions, the results obtained here corroborated with some previous results one can observe the richness of the network dynamic behavior.

The richness of the network's nonlinear behavior – CMN/L using as characteristic functions this class of fractions makes it proper to use in various applications. It can be used to model oscillating neurons [2][5][6][7] – in this paper we have seen that the network (CMN/L) response is much richer then a ganglia like network – or can be used in modeling natural processes – processes that are involving the process of spreading. Depending on the type, the class of the characteristic function and depending to the information aggregation process we can model a variation of the process that we study or even a new class/ another process. Biological processes like cells-gene interactions were simulated with coupled map lattices due to their features [16][17].

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