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Evaluating Weapons by Ranking Fuzzy Numbers Based on Relative Distance

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Abstract: In this paper, we propose two new approaches for ranking fuzzy numbers, one is ranking fuzzy numbers based on relative distance; the other is ranking fuzzy numbers by Boltzmann entropy. Our main study is concentrated on ranking fuzzy numbers by calculating the relative distance, which is calculated the distance between fuzzy numbers, namely, the Relative Distance (RD). We compute the distance for each fuzzy number to its fuzzy maximum set $\tilde{r}_{\max}$ and its fuzzy minimum set $\tilde{r}_{\min}$, where $\tilde{r}_{\max}$ and $\tilde{r}_{\min}$ are two ideal fuzzy sets. The relative distance is defined as $RD = d(\tilde{A}_i, \tilde{r}_{\min}) / d(\tilde{A}_i, \tilde{r}_{\max})$, when the distance between $\tilde{A}_i$ to $\tilde{r}_{\max}$ is small and the distance between $\tilde{A}_i$ to $\tilde{r}_{\min}$ is large, then the RD value with a higher score is considered better. At last, we construct a numerical example for selecting the best attack helicopter to illustrate our proposed method.
Keywords: Military applications, Ranking fuzzy numbers, Relative Distance (RD), Boltzmann Entropy, Fuzzy Multiple Attribute Decision-Making (MADM).

1. Introduction

Traditionally, the general evaluation of criteria is sometimes vagueness, and can not be shown by a crisp number. So, we can describe it by fuzzy method and use fuzzy mathematics to solve it. In many fuzzy multiple attribute decision making (MADM) problems, since many decision makers can not represent their opinion very well, and sometimes the opinions have some kinds of fuzziness. Due to this point, many scholars think that the use of fuzzy numbers can represent their opinion more efficiently, and how to aggregation these fuzzy numbers is an important issue. So sometimes the final scores of alternatives are represented in term of fuzzy numbers. In order to choose a best alternative, we need a method for building a well-defined total ordering of the fuzzy numbers. Many other fuzzy methods and models have been suggested to solve the MADM problem. They differ by their assumptions concerning the input data and by the measures used for aggregation and ranking. Also, they concentrate either on the first step (aggregation of ratings), or the second step (ranking), or both. Many ranking methods have been proposed so far. However, there is yet no method that cans always gives a satisfactory solution to every situation. Some of these methods are counterintuitive, not discriminating; some use only the local information of fuzzy values; some produce different rankings for the same situation. Obviously, all of them have advantages and disadvantages.

In practical use, ranking fuzzy numbers is very important. For example, to come true the concept of optimum or the best choice is completely based on ranking or comparison. Therefore, how do we set the ranking fuzzy numbers have been one of the main problems. The concept of fuzzy numbers is presented by Jain [13] and Dubois and Prade [10]. To resolve the task of comparing fuzzy numbers, many authors have proposed fuzzy ranking methods that yield a totally ordered set or ranking. These methods range from the trivial to the complex, from including one fuzzy number attribute to including many fuzzy number attributes. A review and comparison of these existing methods can be found in [5, 20, 28]. The ranking methods are classified into four major classes according to Chen and Hwang [5], which is listed in the following.

(1) Preference relation

- **a)** Degree of optimality (Such as Baas and Kwakernaak [2], Wastson et al.[28] and Baldwin and Guild[3])
- **b)** Hamming distance (Such as Yager [29], Kerre [17], Nakamura [25], and Kolodzijezyk [18])
- **c)** α-cut (Such as Adamo [1], Buckley and Chanas [4], and Mabuchi [23])
- **d)** Comparison function (Such as Dubois and Prade [12], Tsukamoto et al. [27], and Delgado et al. [10]).

(2) Fuzzy mean and spread

Probability distribution (Such as Dubois and Prade [12], Tsukamoto et al. [27], and Del-gado et al. [10]).

(3) Fuzzy scoring

- a) Proportion to optimal (Such as Lee and Li [21])
- b) Left/right scores (Such as Jain [14-15], Chen[5], and Chen And Hwang [7])
- c) Centroid index (Such as Yager [30], Murakami et al.[24])

Area measurement (Such as Liou and Wang [22], Yager [31]).

(4) Linguistic expression

- a) Intuition (Such as Efstathiou and Tong [13])
- b) Linguistic approximation (Such as Tong and Bonissone [26])

For overcoming above problems, in this paper we propose a new method for ranking fuzzy number based on distance between fuzzy numbers, namely, the **Relative Distance (RD),** we compute the distance of each fuzzy number to $\tilde{r}_{\max}$ and $\tilde{r}_{\min}$, where $\tilde{r}_{\max}$ and $\tilde{r}_{\min}$ are two ideal fuzzy sets and its relative value is defined as $RD = d(\tilde{A}_i, \tilde{r}_{\min}) / d(\tilde{A}_i, \tilde{r}_{\max})$. When the distance between $\tilde{A}_i$ to $\tilde{r}_{\max}$ is small and the distance between $\tilde{A}_i$ to $\tilde{r}_{\min}$ is large, then the value of RV with a higher score is considered better. We can see some examples in section 4.

In this paper, we first present basic concepts and definitions of fuzzy arithmetic, and we introduce another new concept called Boltzmann entropy. Due to Lee and

Li [21] point out that human intuition would favor a fuzzy number with the following characteristic: higher mean value and, at the same time, lower spread. When the fuzzy numbers have the same mean values, we can calculate their fuzziness by Boltzmann entropy to be an index in measure of fuzziness. At the same time, there are another two methods using Hamming distance for ranking fuzzy number between fuzzy sets: Yager‘s [29] and Kerre’s [17], they compute the distance between fuzzy numbers, but both of them have such a condition they can‘t distinguish. Such as that Yager can’t deal with crisp number and Kerre can’t deal with small area measurement. But, in this paper, we propose a new method with two characteristics ($\tilde{r}_{\max}$ and $\tilde{r}_{\min}$) that can solve their shortcomings in deal with crisp number in Yager‘s method and the small area measurement in Kerre’s method. At last, we present some numerical examples of above conditions and a numerical example to illustrate our method, and hopefully, we can build a well ranking fuzzy number for decision-making.

2. Arithmetic operations of fuzzy numbers

Fuzzy arithmetic is based on two properties of fuzzy number [18]: (i) each fuzzy set, and thus also each fuzzy number, can fully and uniquely be represent by its α-cuts; and (ii)α-cuts of each fuzzy number are closed intervals of real numbers for all $\alpha \in [0,1]$. These properties enable us to define arithmetic operations on fuzzy numbers in terms of arithmetic operation on their α-cuts. In addition, fuzzy numbers are convex, will be used throughout.

Definition 2.1 [8]

Let A be a fuzzy set on real number R, $\forall \alpha \in [0,1]$, A_α is a closed interval, i.e., A_α = [a_α, b_α], then called A is a Fuzzy real number, briefly, Fuzzy number, it can denotes as $\tilde{A}$

Definition 2.2 [32]

A fuzzy number is a continuous fuzzy set of universe R with convex membership function $\mu_{\tilde{A}}(x)$ with the following requirement:

$$\max_{x \in R} \mu_{\tilde{A}}(x) = 1 \tag{1}$$

Definition 2.3 [16]

A triangular fuzzy number (denoted as $\tilde{A}$) can be defined as a triplet $\tilde{A}$ =(a, b, c), which is shown as Fig. 1. The membership function is define as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 - \dfrac{x-b}{c-b} & b \le x \le c \\ 0 & \mathrm{x > c} \end{cases} \tag{2}$$

we can characterize the triangular fuzzy number as :

$$\forall \alpha \in [0,1] \qquad \tilde{A}_{\alpha} = [a^{\alpha}, c^{\alpha}] = [(b-a)\alpha + a, -(c-b)\alpha + c] \qquad (3)$$

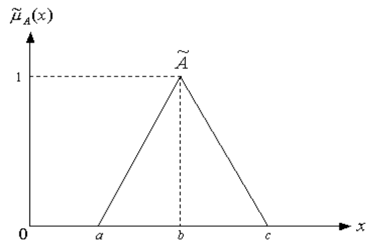

Fig. 1 – The triangular fuzzy number $\tilde{A}$

Definition 2.4 [8-9]
Let $\tilde{A}$ and $\tilde{B}$ be two positive fuzzy numbers with the triplet (a_1, a_2, a_3) and (b_1, b_2, b_3) then the operations of triangular fuzzy numbers are expressed as:

$$\begin{aligned} (a_1, a_2, a_3) \oplus (b_1, b_2, b_3) &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ (a_1, a_2, a_3) \otimes (b_1, b_2, b_3) &= (a_1 b_1, a_2 b_2, a_3 b_3) \end{aligned} \qquad (4)$$

where $\oplus$ and $\otimes$ represent fuzzy number addition and fuzzy number multiplication.

In this paper, the computational technique is based on the following fuzzy numbers defined in Table 1. Each characteristic function is defined by three parameters of the symmetric triangular fuzzy number, the left point, middle point and right point of the range over which function is defined. The meaning of relative strength for fuzzy ratio scales $\tilde{1}, \tilde{3}, \tilde{5}, \tilde{7}$, and $\tilde{9}$ is shown in Table 2.

Table 1

Characteristic function of the fuzzy numbers

Fuzzy number	Characteristic (Membership) function
$\tilde{1}$	(1, 1, 3)
$\tilde{x}$	(x-2, x, x+2) for x = 3, 5, 7
$\tilde{9}$	(7, 9, 9)

Table 2

The meaning of relative strength for fuzzy scales

Intensity of importance	The means of fuzzy numbers for linguistic represents
$\tilde{1}$	Almost equal importance
$\tilde{3}$	Moderate importance of one over another
$\tilde{5}$	Strong importance
$\tilde{7}$	Very Strong importance
$\tilde{9}$	Extreme importance

3. The new approach for ranking fuzzy numbers

In this section, we will propose two methods for ranking fuzzy numbers, one is ranking fuzzy numbers based on calculating relative distance value; the other is ranking fuzzy number based on Boltzmann entropy.

3.1. Ranking fuzzy numbers based on calculating relative distance

From Kaufmann and Gupta [16], we will review some important concepts of distance in the followings.

Definition 3.1: fuzzy maximum set $\tilde{r}_{max}$ and fuzzy minimum set $\tilde{r}_{min}$

(A) A fuzzy maximum set $\tilde{r}_{max}$ is defined as

$$\{(x, \mu_{max}(x))\}, \text{ where } \mu_{max}(x) = \frac{1}{\beta_2 - \beta_1}(x - \beta_1),\ x \in [\beta_1, \beta_2], \tag{5}$$

and $[\beta_1, \beta_2]$ is a interval satisfy that every interval of fuzzy set with $\alpha = 0$ level is a subset of $[\beta_1, \beta_2]$.

(B) Similarly, a fuzzy minimum set $\tilde{r}_{min}$ is defined as

$$\{(x, \mu_{max}(x))\}, \text{ where } \mu_{min}(x) = \frac{-1}{\beta_2 - \beta_1}(x - \beta_2),\ x \in [\beta_1, \beta_2], \tag{6}$$

the restrict condition is the same as (A).

Definition 3.2: The concept of distance

Let us consider three intervals of confidence in R, namely,

$$A = (a_1, a_2),\ B = (b_1, b_2),\ C = (c_1, c_2)$$

Any concept of distance must satisfy the following properties:

A numerical function $d(X,Y) \in$ R, $(X,Y) \in E \times E$ is a distance if and only if $\forall X, Y, Z \in E$;

$$(1)\ d(\mathrm{X}, \mathrm{Y}) \geq 0 \tag{7}$$

$$(2)\ (\mathrm{X}=\mathrm{Y}) \Rightarrow (d(\mathrm{X}, \mathrm{Y})=0) \tag{8}$$

$$(3)\ d(\mathrm{X}, \mathrm{Y}) = d(\mathrm{Y}, \mathrm{X}) \tag{9}$$

$$(4)\ d(\mathrm{X}, \mathrm{Z}) \leq d(\mathrm{X}, \mathrm{Y}) * d(\mathrm{Y}, \mathrm{Z}) \tag{10}$$

where $*$ is an operator associated with the notion of distance. This concept of distance is different, of course, from our usual concept of "metric" where we assume that $(X=Y) \Leftrightarrow (d(X, Y)=0)$. If this was not so, in metric we would not be able to find $d(X,Y)=0$ when $X \neq Y$.

To consider the concept of distance to the left, let

$$\Delta_l(A,B) = |a_1 - b_1| \tag{11}$$

Similarly, for the distance to the right, let

$$\Delta_r(A,B) = |a_2 - b_2| \tag{12}$$

We must now check the conditions from equation (7)-(10) for this concept of distance.

$\forall\ A, B, C \subset$ R:

1. $\Delta_l(A,B) \geq 0,$ because $|a_1 - b_1| \geq 0$
2. $(A = B) \Rightarrow \Delta_l(A,B) = 0,$, because $a_1 = b_1 \Rightarrow |a_1 - b_1| = 0$
3. $\Delta_l(A,B) = \Delta_l(B,A),$, because $|a_1 - b_1| = |b_1 - a_1|$
4. $\Delta_l(A,C) \leq \Delta_l(A,B) + \Delta_l(B,C),$, because $|a_1 - c_1| \leq |a_1 - b_1| + |b_1 - c_1|$

The proof for Δ_r may be carried out in the same way.

Now, we consider the concept of distance $\Delta(A,B)$ as follows:

$\Delta(A,B) = \Delta_l(A,B) + \Delta_r(A,B)$

It is easy to prove the conditions of equation (7)-(10) are also satisfied for this condition.

Suppose now that any interval like $A = (a_1,a_2)$, $B = (b_1,b_2)$, and $C = (c_1, c_2)$ is a subset of $[\beta_1, \beta_2] \subset R$. We define

$$\delta(A,B) = \{1/[2(\beta_2 - \beta_1)]\}d(A,B). \tag{13}$$

as the normalized distance. The number 2 in the denominator permits us to write

$$0 \leq \delta(A,B) \leq 1 \tag{14}$$

For example, suppose we have two fuzzy numbers A and B in R as in Fig. 2.
For each level α, we can write

$$\delta(A_\alpha, B_\alpha) = \{1/[2(\beta_2 - \beta_1)]\}d(A_\alpha, B_\alpha), \tag{15}$$

where β_1 and β_1 are given any convenient values in order to surround all $A_\alpha = 0$ and $B_\alpha = 0$.

If we proceed to integrate from α=0 to α=1, we obtain a distance by the summation of distances that satisfies equation (14),

$$\delta(A,B) = \int_{\alpha=0}^{1} \delta(A_\alpha, B_\alpha)d\alpha = 1/2(\beta_2 - \beta_1) \int_{\alpha=0}^{1} \Delta(A_\alpha, B_\alpha)d\alpha$$

$$= 1/2(\beta_2 - \beta_1) \int_{\alpha=0}^{1} \left(\left|a_1^{(\alpha)} - b_1^{(\alpha)}\right| + \left|a_2^{(\alpha)} - b_2^{(\alpha)}\right|\right)d\alpha \tag{16}$$

Equation (16) gives the distance between two fuzzy numbers; it may also be called the dissemblance index of *A* and *B*.

From Kaufmann and Gupta's dissemblance index of *A* and *B* [16], we propose a new method for ranking fuzzy numbers based on calculating relative distance value. What is relative distance value?
We define the relative distance as

$$RD = d(\tilde{A}_i, \tilde{r}_{\min}) / d(\tilde{A}_i, \tilde{r}_{\max}) \qquad (17)$$

where $d(\tilde{A}_i, \tilde{r}_{\min})$ denotes the distance between each $\tilde{A}_i$, $\forall i : i = 1,..., n$ to $\tilde{r}_{\min}$, and $d(\tilde{A}_i, \tilde{r}_{\max})$ represent the distance between each $\tilde{A}_i$, $\forall i : i = 1,..., n$ to $\tilde{r}_{\max}$.
For easy computing, the algorithm of our method can be listed in the following.

(1) Set two ideal fuzzy sets: fuzzy Min $\tilde{r}_{\min}$ and fuzzy Max $\tilde{r}_{\max}$, which satisfy that any interval of fuzzy number is a subset of $[\beta_1\ \beta_2]$, where $[\beta_1\ \beta_2]$ is an interval that can contain any fuzzy number $\tilde{A}_i$
(2) From equation (13), calculate the distance between each $\tilde{A}_i$, $\forall i\ i = 1,..., n$ to $\tilde{r}_{\min}$, *i.e.*, $d(\tilde{A}_i, \tilde{r}_{\min})$, and calculating the distance between each $\tilde{A}_i$, $\forall i\ i = 1,..., n$ to $\tilde{r}_{\max}$, *i.e.*, $d(\tilde{A}_i, \tilde{r}_{\max})$.
(3) Calculate the relative distance $RD = d(\tilde{A}_i, \tilde{r}_{\min})/d(\tilde{A}_i, \tilde{r}_{\max})$ for each fuzzy number by equation (17).
(4) Rank its ordering: The relative distance value is the largest, and the corresponding to fuzzy number is the best ordering.

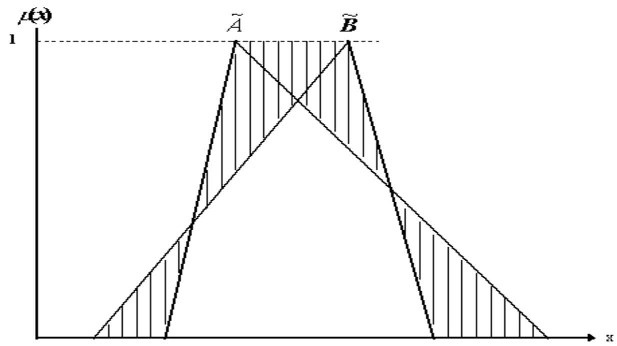

Fig. 2 – The shadow part represent the distance between $\tilde{A}$ and $\tilde{B}$

3.2 Ranking fuzzy number based on Boltzmann entropy

The question of how to measure vagueness or fuzziness has been one of the issues associated with the development of the theory of fuzzy sets. Measures of fuzziness by contrast to fuzzy measures try to indicate the degree of fuzziness of a fuzzy set. Several measures of fuzziness have been proposed in the literature. The Shannon entropy, which is a measure of uncertainty and information formulated in terms of probability theory, is expressed by the function

$$H\left(p(x)\middle|x\in X\right)=-\sum_{x\in X}p(x)\log_2 p(x) \tag{18}$$

where $(p(x)|\ x\in X)$ is a probability distribution on a finite set X.

Due to fuzzy number usually is represented by continuous membership function, the Shannon entropy is connected with its restriction to finite sets. Is this restriction necessary ? It seems that the formula

$$B\left(q(x)\middle|x\in[a,b]\right)=-\int_a^b q(x)\log_2 q(x)dx \tag{19}$$

where q denotes a probability density function on the real interval [a,b], is analogous to formula equation (18) for Shannon entropy and could thus be viewed as an extension of Shannon entropy to the domain of real numbers. Moreover, function B is defined by equation (19) is usually referred to as the Boltzmann entropy. Therefore, we can use the Boltzmann entropy to rank fuzzy numbers with continuous membership function.

In many fuzzy multiple criteria decision making problem, the final scores of alternatives are represented in term of fuzzy numbers. In order to choose a best alternative, we need a method for constructing a crisp total ordering from fuzzy numbers. Many methods for ranking of fuzzy numbers have been suggested. Each method appears to have some advantages as well as disadvantages [18]. In fuzzy multiple criteria decision making problem, many triangular fuzzy numbers can intuitively rank its ordering by drawing its curves. If its ordering can not rank by Figures, we can use many other methods of ranking fuzzy numbers.

From Lee and Li [21] and the concept of statistics, the standard deviation and mean value cannot be the sole basic for comparing two fuzzy numbers, respectively. From Lee and Li’s idea: higher mean value and at the same time lower spread is ranked higher. Therefore, we propose an efficient index, which is, using the Boltzmann entropy to rank fuzzy numbers, its algorithm can be summarized in the following.

(1) Calculate the mean values of fuzzy numbers and compare its mean values, the largest mean is the best ordering for fuzzy number.

(2) If the mean values of fuzzy numbers are same, computing its Boltzmann entropy by equation (19), the smallest weight is the best ordering for fuzzy number.

For example:

Let us consider two fuzzy numbers $\tilde{A}$ (1,4,7) and $\tilde{B}$ (2,4,6).

Due to the two fuzzy numbers have the same mean values equal to 4. Hence, we can use equation (19) to calculate its entropy, which is shown in Table 3.

Table 3

The Boltzmann entropy for fuzzy number $\tilde{A}$ and $\tilde{B}$

	$\tilde{A}$	$\tilde{B}$
ENTROPY	2.164	1.443

From Table 3, its ordering is $\tilde{A} < \tilde{B}$, and from our intuition, $\tilde{B}$ is better than $\tilde{A}$.

4. The example for ranking fuzzy number by RV method

In this section, we give three different types of examples to illustrate our method for ranking fuzzy numbers based on relative distance.

Example 4.1

This example is from Laarhoven and Pedrycz [20]. Let us consider three triangular fuzzy numbers ($\tilde{U}_1$, $\tilde{U}_2$, and $\tilde{U}_3$), and it can be form as follows (see Fig. 3):

$$\tilde{U}_1=(0.2,0.3,0.5),\ \tilde{U}_2=(0.17,0.32,0.58),\ \tilde{U}_3=(0.25,0.4,0.7)$$

(1) The membership function and α-cut interval can be written in the Table 4. We set two ideal fuzzy set: fuzzy Min $\tilde{r}_{\min}$ and fuzzy Max $\tilde{r}_{\max}$, which satisfy that the interval of fuzzy number is a subset of $[\beta_1, \beta_2]$=[0.1,0.8], and its graph can be plotted as Fig. 4.

(2) Calculate the distance between $\tilde{U}_i$ to $\tilde{r}_{\max}$ and $\tilde{r}_{\min}$ by equation (13), its results are listed in Table 5.

From Table 5, we obtain its ordering is $\tilde{U}_3 > \tilde{U}_2 > \tilde{U}_1$.

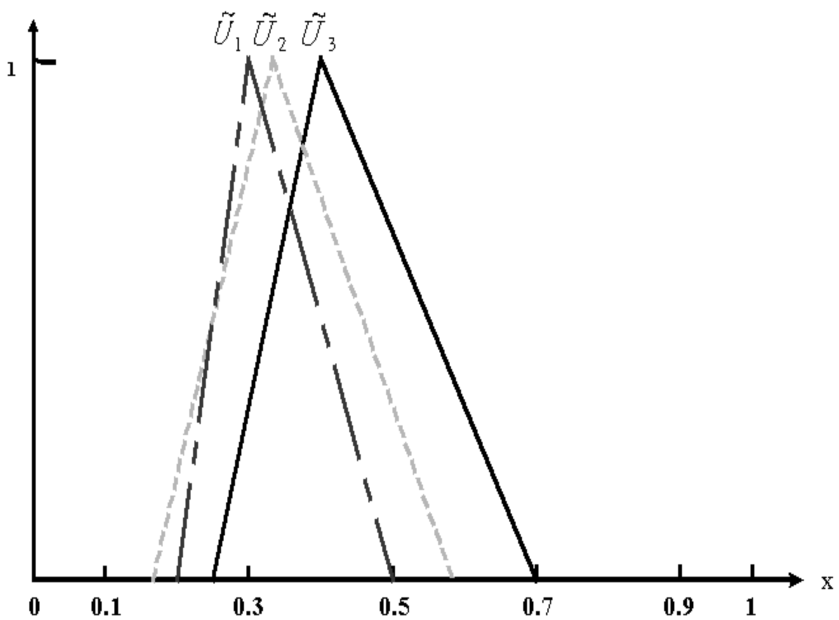

Fig. 3 – Three triangular fuzzy numbers $\tilde{U}_1$, $\tilde{U}_2$, and $\tilde{U}_3$

Table 4

The membership function and α-cut interval for three fuzzy numbers

	Membership function $f_{\tilde{U}_i}$	α-cut interval $\tilde{U}_i^{\alpha}$
$\tilde{U}_1$	$\begin{cases} \frac{x-0.2}{0.1} & 0.2 \le x < 0.3 \\ 1 & x = 0.3 \\ \frac{0.5-x}{0.2} & 0.3 < x \le 0.5 \end{cases}$	$\tilde{U}_1^{\alpha} = [0.2+0.1\alpha, 0.5-0.2\alpha]$
$\tilde{U}_2$	$\begin{cases} \frac{x-0.25}{0.15} & 0.25 \le x < 0.4 \\ 1 & x = 0.4 \\ \frac{0.7-x}{0.3} & 0.4 < x \le 0.7 \end{cases}$	$\tilde{U}_2^{\alpha} = [0.17+0.15\alpha, 0.58-0.14\alpha]$
$\tilde{U}_3$	$\begin{cases} \frac{x-0.17}{0.15} & 0.17 \le x < 0.32 \\ 1 & x = 0.32 \\ \frac{0.58-x}{0.26} & 0.32 < x \le 0.58 \end{cases}$	$\tilde{U}_3^{\alpha} = [0.25+0.15\alpha, 0.7-0.3\alpha]$

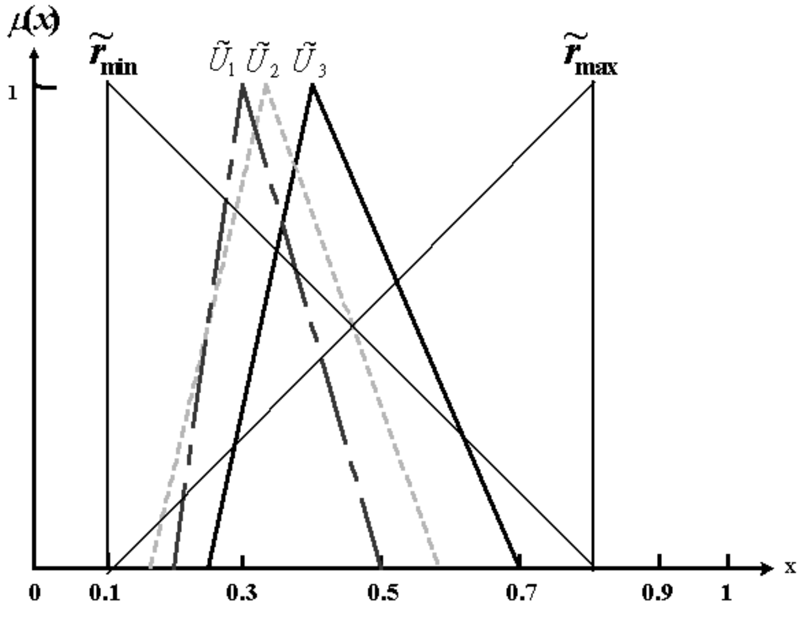

Fig. 4 – Three triangular fuzzy numbers and given $\tilde{r}_{\min}$ and $\tilde{r}_{\max}$

Table 5

The RV values and the distance between $\tilde{U}_i$ to $\tilde{r}_{\max}$ and $\tilde{r}_{\min}$

Fuzzy number	$\tilde{r}_{\max}$	$\tilde{r}_{\min}$	Relative distance
$\tilde{U}_1$	0.44	0.2	0.455
$\tilde{U}_2$	0.402	0.207	0.515
$\tilde{U}_3$	0.297	0.25	0.842

Example 4.2

Let us consider two fuzzy numbers $\tilde{A}$ = (0.1, 0.2, 0.3) and $\tilde{B}$ = (0.35, 0.5, 0.8), the example is taken from Kerre[17].

(1) Set $[\beta_1, \beta_2]$= [0,0.9], and its graph can be plotted as Fig. 5.

(2) Calculate the distance between $\tilde{A}$ and $\tilde{B}$ to $\tilde{r}_{\max}$ and $\tilde{r}_{\min}$ by equation (13), respectively, its results are listed in Table 6.

Table 6

The RV values and the distance between fuzzy number to $\tilde{r}_{\max}$ and $\tilde{r}_{\min}$

Fuzzy number	$\tilde{r}_{\max}$	$\tilde{r}_{\min}$	Relative distance
$\tilde{A}$	0.962	0.4	0.416
$\tilde{B}$	0.438	0.642	1.466

From Table 6, we obtain its ordering is $\tilde{B} > \tilde{A}$.

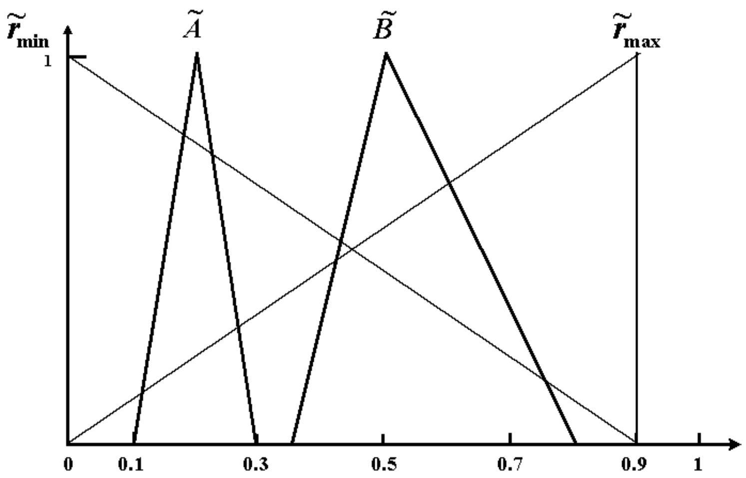

Fig. 5 – Two triangular fuzzy numbers and given $\tilde{r}_{\min}$ and $\tilde{r}_{\max}$

Example 4.3

Consider a crisp number $\tilde{A}$ =(0.5,0.5,0.5), $\tilde{B}$ =(0.1,0.2,0.3) and another crisp number $\tilde{C}$ =(0.15,0.15,0.15), let $\tilde{r}_{\max}$ =(0,0.6,0.6), and $\tilde{r}_{\min}$ =(0,0,0.6)

(1) Set $[\beta_1,\beta_2]$ = [0,0.6], and its graph can be plotted as Fig. 6.

(2) Calculate the distance between $\tilde{A}$ $\tilde{B}$ and $\tilde{C}$ to $\tilde{r}_{\max}$ and $\tilde{r}_{\min}$ by equation (13), respectively, its results are listed in Table 7.

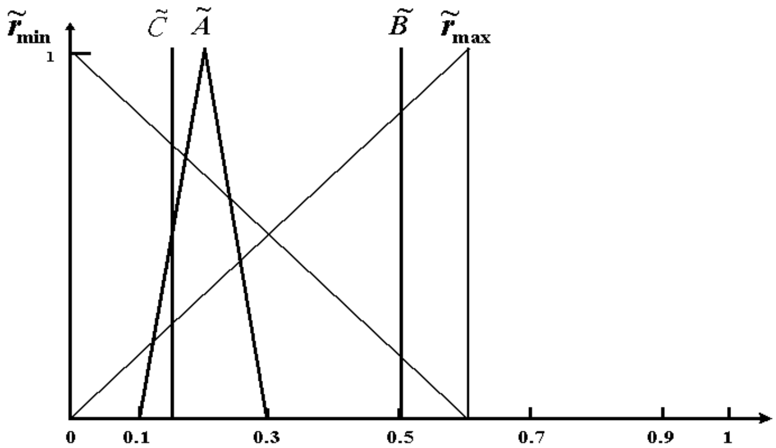

Fig. 6 – Three triangular fuzzy numbers and given $\tilde{r}_{\min}$ and $\tilde{r}_{\max}$

Table 7

The RV values and the distance between fuzzy number to $\tilde{r}_{\max}$ and $\tilde{r}_{\min}$

Fuzzy number	$\tilde{r}_{\max}$	$\tilde{r}_{\min}$	Relative distance
$\tilde{A}$	0.317	0.717	2.262
$\tilde{B}$	0.52	0.28	0.538
$\tilde{C}$	0.638	0.338	0.53

From Table 7, we obtain its ordering is $\tilde{A} > \tilde{B} > \tilde{C}$.

5. The algorithm for application in decision making

In this section, we present an algorithm for evaluating weapon systems by ranking fuzzy numbers based on relative distance between fuzzy numbers. The computational procedure of this decision making methodology can be listed in the following:

Step1: Construct fuzzy judgement matrix

Use fuzzy number to indicate the relative contribution or impact of each element on each governing objective or criterion in the adjacent upper level. For each criterion, construct a fuzzy judgement vector. Then all fuzzy judgement vectors among each alternative are composed fuzzy judgement matrix. In such matrix of fuzzy number, the elements (fuzzy numbers) are through comparison of the performance scores in the same criterion. Then, the fuzzy judgement matrix can be structured by all fuzzy judgement vectors.

Step2: Multiplying the fuzzy judgement matrix with the corresponding fuzzy weight vector. *i.e.*,

$$\tilde{R}=\tilde{A}\otimes\tilde{W}^T=\begin{bmatrix}\tilde{a}_{11} & \tilde{a}_{12} & \ldots & \tilde{a}_{1n}\\ \tilde{a}_{21} & \tilde{a}_{22} & \ldots & \tilde{a}_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ \tilde{a}_{m1} & \tilde{a}_{m2} & \ldots & \tilde{a}_{mn}\end{bmatrix}\otimes\begin{bmatrix}\tilde{W}_1\\ \tilde{W}_2\\ \ldots\\ \tilde{W}_n\end{bmatrix}=\begin{bmatrix}\tilde{a}_{11}\otimes\tilde{W}_1\oplus\tilde{a}_{12}\otimes\tilde{W}_2\oplus\ldots\oplus\tilde{a}_{1n}\otimes\tilde{W}_n\\ \tilde{a}_{21}\otimes\tilde{W}_1\oplus\tilde{a}_{22}\otimes\tilde{W}_2\oplus\ldots\oplus\tilde{a}_{2n}\otimes\tilde{W}_n\\ \ldots\\ \tilde{a}_{m1}\otimes\tilde{W}_1\oplus\tilde{a}_{m2}\otimes\tilde{W}_2\oplus\ldots\oplus\tilde{a}_{mn}\otimes\tilde{W}_n\end{bmatrix}=\begin{bmatrix}\tilde{r}_1\\ \tilde{r}_2\\ \ldots\\ \tilde{r}_n\end{bmatrix}$$

Step3: Ranking fuzzy number

Many triangular fuzzy numbers can intuitively rank its ordering by drawing its curves. If its ordering can not rank by intuition ranking method, we can rank fuzzy number $\tilde{r}_1,\tilde{r}_2,\ldots,\tilde{r}_n$ by our method to determine the best weapon system. i.e.,

(1) We set two ideal fuzzy set: fuzzy Min $\tilde{r}_{\min}$ and fuzzy Max $\tilde{r}_{\max}$, which satisfy that any interval of fuzzy number is a subset of $[\beta_1\ \beta_2]$ where $[\beta_1\ \beta_2]$ is a interval that can contain any fuzzy number $\tilde{A}_i$, and then

(2) Calculate the distance between each $\tilde{A}_i$, $\forall i$ i = 1,..., n to $\tilde{r}_{\min}$, then we choice the larger distance as our decision. Similarly, the distance between each $\tilde{A}_i$, $\forall i$ i = 1,..., n to $\tilde{r}_{\max}$, then

(3) Calculate all RV = $d(\tilde{A}_i,\tilde{r}_{\min})/d(\tilde{A}_i,\tilde{r}_{\max})$,

(4) The larger for relative distance is the better ordering for corresponding to fuzzy number.

6. Selecting the best attack helicopter by our method

In this section, for illustrating our proposed method, we have constructed an evaluation model for three types of Attack Helicopter [9]. The evaluation is based on five criteria: technological advance (C1), logistic capability (C2), armament (C3), avionics (C4), and subsisting ability (C5).

Their criteria, sub-criteria and computational procedure are detailed in the following.

(1) Structure the hierarchical figure of attack helicopters as Fig. 7.

(2) From the Step 1 in section 5, find their degree of membership function for each system with respect to each item as Table 8, then compute their total scores. We use fuzzy number $\tilde{1}$, $\tilde{3}$, $\tilde{5}$, $\tilde{7}$, and $\tilde{9}$ to indicate the relative contribution or impact of each element on each governing objective or criterion in the adjacent upper level. For each criterion, the elements (fuzzy numbers) are through

comparison of the performance scores in the same criterion. In the same way, all results of criteria and data are listed in Tables 8-15.

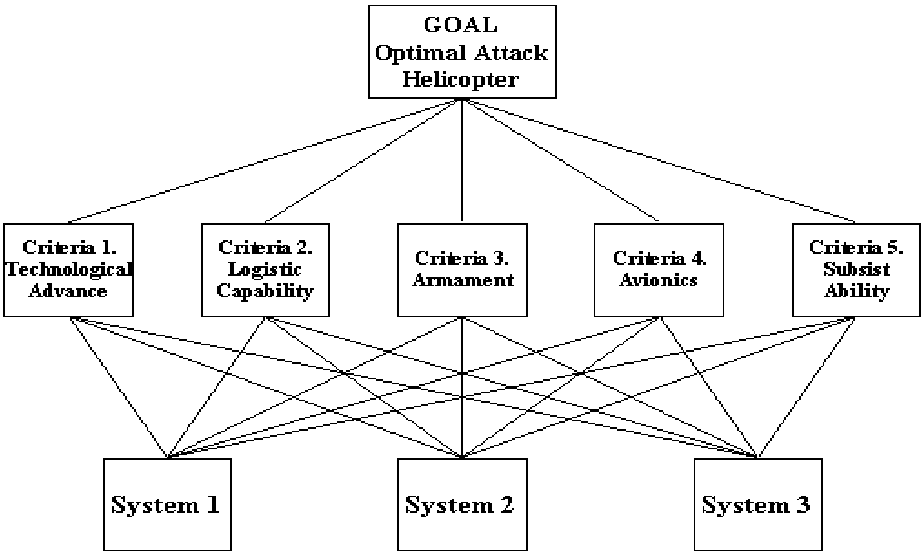

Fig. 7 – The structure model of evaluating three Attack Helicopters

I. Technological advance

Table 8

Technological advance data for three Attack Helicopter and its judgment criteria

	Item	S1	S2	S3	Membership function
1	Turbo-shafts (kw)	1633x2	1265x2	1285x2	$\mu_{tu}=\begin{cases}(x-1100)/300 & 1100\le x\le 1500\\ 1 & 1500\le x\end{cases}$
2	Weight empty (kg)	7000	5092	4634	$\mu_{we}=\begin{cases}(8000-x)/4000 & 4000\le x\le 8000\\ 1 & x\le 4000\end{cases}$
3	Max level speed (km/h)	300	293	282	$\mu_{ml}=\begin{cases}(x-250)/70 & 320\le x\le 250\\ 1 & 320\le x\end{cases}$
4	Max disc loading (kg/m2)	49	56.69	39.80	$\mu_{md}=\begin{cases}(x-30)/40 & 30\le x\le 70\\ 1 & 70\le x\end{cases}$
5	Max disc loading (kg/m2)	5800	6400	4270	$\mu_{sc}=\begin{cases}(x-4000)/3000 & 70000\le x\le 4000\\ 1 & 7000\le x\end{cases}$
6	Service ceiling (m)	460	482	507	$\mu_{mr}=\begin{cases}(x-400)/200 & 400\le x\le 600\\ 1 & 600\le x\end{cases}$
7	Maximal range standard fuel (km)	2h	3h9min	2h	$\mu_{em}=\begin{cases}0.5 & 2h\le x<3h\\ 1 & 3h\end{cases}$
8	Endurance with Maximal fuel	+3/-0.5	+3.5/-0.5	+2.5/-0.5	$\mu^{+}_{gl}=\begin{cases}(x^{+}-2)/2 & 2\le x^{+}\le 4\\ 1 & 4\le x^{+}\end{cases}$
9	g-limits	90	93	92	$\mu_{mc}=\begin{cases}x/10 & 0\le x<100\\ 1 & 100\le x\end{cases}$

Table 9

Technological advance scores derived form Table 8

	Item	S1	S2	S3
1	Turbo-shafts (kw)	1	0.41	0.46
2	Weight empty (kg)	0.25	0.72	0.84
3	Max level speed (km/h)	0.71	0.61	0.46
4	Max disc loading (kg/m2)	0.49	0.67	0.25
5	Max disc loading (kg/m2)	0.6	0.8	0.09
6	Service ceiling (m)	0.3	0.41	0.54
7	Maximal range standard fuel (km)	0.5	1	0.5
8	Endurance with Maximal fuel	0.5	1	0.25
9	g-limits	0.9	0.93	0.92
Total		5.25	6.55	4.31
Fuzzy number		$\tilde{5}$	$\tilde{9}$	$\tilde{1}$

II. Logistic capability

Table 10

Expert evaluations of Logistic capability represented by linguistic terms

	Item	S1	S2	S3
1	Reliability	fair	good	good
2	Maintenance ability	very good	good	good
3	Convey	fair	very good	good
4	Economics	very good	good	very good
5	Flexibility for selecting weapon	good	very good	good

Table 11

Logistic scores derived form Table 10

	Item	S1	S2	S3
1	Reliability	fair	good	good
2	Maintenance ability	very good	good	good
3	Convey	fair	very good	good
4	Economics	very good	good	very good
5	Flexibility for selecting weapon	good	very good	good
Total		5.25	6.55	4.31
Fuzzy number		$\tilde{1}$	$\tilde{5}$	$\tilde{3}$

III. Armament

Table 12

Armament data for three Attack Helicopter and its judgment criteria

Item		S1	S2	S3	Membership function
Gun	Caliber (mm)	30	30	20	$\mu_{gc}=\begin{cases}0.5 & 20\le x<30\\ 1 & 30=x\end{cases}$
	Firing rate (r/m)	900	625	650	$\mu_{fr}=\begin{cases}(x-500)/500 & 500\le x\le 1000\\ 1 & 1000\le x\end{cases}$
	Feed	300	1200	750	$\mu_{gf}=\begin{cases}(x-200)/1000 & 200\le x<1200\\ 1 & 1200\le x\end{cases}$
Anti-tank missiles	Feed	16	16	8	$\mu_{af}=\begin{cases}x/16 & 0\le x<16\\ 1 & 16\le x\end{cases}$
	Firing range (km)	5	8	8	$\mu_{ar}=\begin{cases}x/10 & 0\le x<10\\ 1 & 10\le x\end{cases}$
	Firing accuracy (%)	80	76	87	$\mu_{aa}=\begin{cases}x/100 & 0\le x<100\\ 1 & 100\le x\end{cases}$
air-to-air missiles	Feed	8	4	4	$\mu_{ar}=\begin{cases}x/10 & 0\le x<10\\ 1 & 10\le x\end{cases}$
	Firing accuracy (%)	85	90	50	$\mu_{ia}=\begin{cases}x/100 & 0\le x<100\\ 1 & 100\le x\end{cases}$
Rockets	Feed	20x4	19x4	19x4	$\mu_{rf}=\begin{cases}x/100 & 0\le x<100\\ 1 & 100\le x\end{cases}$
	Caliber (mm)	70	70	70	$\mu_{rf}=\begin{cases}x/100 & 0\le x<100\\ 1 & 100\le x\end{cases}$

Table 13

Armament scores derived form Table 12

	Item	S1	S2	S3
Gun	Caliber (mm)	1	1	0.5
	Firing rate (r/m)	0.8	0.25	0.3
	Feed	0.1	1	0.75
Anti-tank missiles	Feed	1	1	0.75
	Firing range (km)	0.5	0.8	0.8
	Firing accuracy (%)	0.8	0.76	0.87
air-to-air missiles	Feed	0.8	0.4	0.4
	Firing accuracy (%)	0.85	0.9	0.5
Rockets	Feed	0.8	0.76	0.76
	Caliber (mm)	0.7	0.7	0.7
Total		7.35	7.57	6.08
Fuzzy number		$\tilde{5}$	$\tilde{7}$	$\tilde{1}$

IV. Avionics

Table 14

Avionics scores

	Item	S1	S2	S3
1	pilot night vision system	0.5	1	0.25
2	target acquisition and designation system	0.5	1	0.5
3	integrate system	0.25	1	0.5
4	global positioning system	0.5	1	0.5
Total		5.25	1.75	4
Fuzzy number		$\tilde{1}$	$\tilde{9}$	$\tilde{1}$

V. Subsisting ability

Table 15

Subsisting ability scores

	Item	S1	S2	S3
1	Armor-protection	0.75	0.75	0.75
2	Counter-detected	0.75	1	0.5
3	Pilot-protested	1	0.75	0.5
4	Noise	1	1	0.5
5	N.B.S. protection	1	0.5	0.5
Total		4.5	4	2.75
Fuzzy number		$\tilde{7}$	$\tilde{5}$	$\tilde{1}$

(3) From Table 8-15, The fuzzy number are through comparison of the performance scores in the same criterion, Then all fuzzy number among each Attack Helicopter and the corresponding criteria are composed fuzzy judgement matrix,

$$\begin{array}{c} \\ \tilde{A} = \begin{array}{c} S1 \\ S2 \\ S3 \end{array} \end{array} \begin{array}{c} \begin{array}{ccccc} C1 & C2 & C3 & C4 & C5 \end{array} \\ \begin{bmatrix} \tilde{5} & \tilde{1} & \tilde{5} & \tilde{1} & \tilde{7} \\ \tilde{9} & \tilde{5} & \tilde{7} & \tilde{9} & \tilde{5} \\ \tilde{1} & \tilde{3} & \tilde{1} & \tilde{1} & \tilde{1} \end{bmatrix} \end{array}$$

(4) Consulting the experts' opinions, we obtain the degree of importance for the criterion's weight. The ordering is technological advance (C1), logistic capability (C2), armament (C3), avionics (C4), and subsisting ability (C5), which is represented by a fuzzy weight vector.

$$\tilde{W} = [\ \tilde{9}\quad \tilde{7}\quad \tilde{5}\quad \tilde{3}\quad \tilde{1}\ ]$$

(5) Multiplying the fuzzy judgement matrix with the corresponding to fuzzy weight vector, that is

$$\tilde{R} = \tilde{A} \otimes \tilde{W}^T = \begin{bmatrix} \tilde{5} & \tilde{1} & \tilde{5} & \tilde{1} & \tilde{7} \\ \tilde{9} & \tilde{5} & \tilde{7} & \tilde{9} & \tilde{5} \\ \tilde{1} & \tilde{3} & \tilde{1} & \tilde{1} & \tilde{1} \end{bmatrix} \otimes \begin{bmatrix} \tilde{9} \\ \tilde{7} \\ \tilde{5} \\ \tilde{3} \\ \tilde{1} \end{bmatrix} = \begin{bmatrix} (41, & 87, & 181) \\ (89, & 169, & 279) \\ (17, & 39, & 117) \end{bmatrix}$$

(6) Ranking fuzzy number by RD method

Use Equation (13) to compute the distance between fuzzy numbers to $\tilde{r}_{\max}$ and $\tilde{r}_{\min}$, its results are shown in Table 16.

Table 16

The results for evaluating three attack helicopter

	$\tilde{r}_{\max}$	$\tilde{r}_{\min}$	RV
S1	0.409	0.198	0.484
S2	0.19	0.38	2
S3	0.563	0.174	0.309

From Table 16, the ordering is S2>S1>S3. Therefore, the system 2 is the best attack helicopter.

7. Conclusions

Many aspects of fuzzy set theory applications require the comparison of fuzzy numbers. Our study proposed two new approaches for ranking fuzzy numbers, one is ranking fuzzy numbers based on relative distance; the other is ranking fuzzy number by Boltzmann entropy. Our main study is concentrated on ranking fuzzy numbers by calculating the relative distance, which is calculated the distance between fuzzy numbers, namely, the Relative Distance (RD). This new method with two characteristics ($\tilde{r}_{\max}$ and $\tilde{r}_{\min}$) that can solve Yager's [29] and Kerre's [17] shortcomings in deal with crisp number in Yager's method and the small area measurement in Kerre's method. Moreover, when the fuzzy numbers have the same mean values, we can calculate their fuzziness by Boltzmann entropy to be an index in measure of fuzziness. We have also constructed a numerical example for selecting attack helicopter to illustrate proposed method, and hopefully can build a generalized method for MADM.

Future research should concentrate on developing a ranking process of fuzzy number, which can prevent the information loss and the problem of ranking reversal. Due to the subjective viewpoint of decision makers, there should also be some ranking methods to make the decisions more flexibile in the decision making process. Besides, the proposed methods can apply to rank the linguistic quantities in the fuzzy rules, MADM, or other application of fuzzy numbers.

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